PERGAMON

International Journal of Solids and Structures 38 (2001) 3487-3548

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Elastic shadow flow and its theoretical implications for inelastic solids

Peter A. Dashner*

Department of Mechanical Engineering, California State Polytechnic University, Pomona, CA, 91768, U.S.A.

Received 12 January 2000; in revised form 10 August 2000

Abstract

Microstructural considerations have led to a constitutive format for the modeling of inelastic solids which does not make use of a fixed material reference in the usual sense. A principal motivation for this 'Eulerian-type' theoretical structure stems from a belief that large deformation inelastic flow phenomena (such as that encountered in certain metal forming processes) will one day prove amenable to numerical techniques of the sort common to the field of fluid mechanics. Unfortunately, virtually all established theoretical results of a general nature have been developed in a Lagrangian or referential context, and are not readily adapted to this new format. Here, this difficulty is partially overcome through the introduction of an instantaneous local reference corresponding to a material element's so called 'shadow' (elastically unstretched) configuration. A related state variable transformation and the definition of a shadow frame time derivative (analogous to the corotational time derivative) are then shown to facilitate a far more tractable theoretical reformulation. Throughout, three equivalent variants of this general theory are presented, one expressed in terms of a gradient type 'cell placement tensor', a second involving symmetric elastic stretch and orthogonal cell orientation tensors, and a third in which the aforementioned elastic stretch is replaced by the elastic (natural) log-strain tensor. In each case, the theoretical simplifications appropriate for various types of materials are enumerated. It is noteworthy that the latter two forms are instantly specialized for the most common class of structurally isotropic materials by merely dropping dependence on the cell orientation tensor. In all forms, basic thermodynamic considerations (second law) result in general expressions for true stress response in terms of energy derivatives, the rate of mechanical dissipation, and to necessary conditions (in the form of constitutive inequalities) for Il'iushin stability. Finally, the specific circumstances under which these inequalities reduce to the familiar 'Drucker-like' forms are identified. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Continua with microstructure; Finite deformation; Elastoplasticity

1. Introduction

In (Dashner, 1986a, b), the efficacy of the classical Green-Naghdi theory (Green *et al.*, 1965) for the modeling of inelastic solids subjected to large strain and deformation was critically assessed, and an alternative proposed. The first issue concerned the imposition of a rotational invariance requirement on

* *E-mail address:* padashner@csupomona.edu (P.A. Dashner)

0020-7683/01/$ - see front matter
PII: S0020-7683(00)00312-7

the so-called "intermediate unstressed configuration." This requirement was shown to place severe, and unrealistic, restrictions on the class of anisotropic materials which can be modeled. In fact, it was demonstrated that this classical theory, based on the state descriptors $(\mathbf{E}, \mathbf{E}_p, \kappa)$, could not model a single anisotropic crystal whose primary inelastic deformation mechanism was that of slip-shear. More generally, it was concluded that this theory can model only those materials whose fundamental anisotropic bond (lattice) structure evolves (during plastic deformation) in a "path-independent" fashion. This led to the assertion that, unless and until the existence of such materials can be demonstrated, its viability outside of the class of structurally isotropic materials is open to question.

In response to this, Casey (1987) conceded that, indeed, an additional orthogonal orientation or "corotation" tensormust be added to the fundamental set of state descriptors $(\mathbf{E}, \mathbf{E}_p)$ for the proper description of anisotropic solids for "which microstructural concepts play an essential role." The addition of this new tensor state variable, along with its attendant evolution equation, must be regarded as a significant theoretical complication - one which has received little attention in the literature to date.

Apart from this, the advisability of a referential (Lagrangian) formulation, in terms of a fixed reference configuration, was also called into question. This was based on the contention that extra geometric variables are required to describe all of the physically relevant configurations as they evolve relative to the chosen reference. Arguing for the essential *path-dependence* of the plastic deformation mechanism, it was asserted[1] in (Dashner, 1986a) that the accumulated plastic deformation (relative to some chosen reference) has no relevance to the characterization of a deformed element's physical state. These arguments were presented in support of a newer formalism for inelastic modeling founded on the assumption that a single 'gradient-type' measure placing the elastically deformed "characteristic lattice cell" in the current configuration, together with a set of "path-dependent" tensor variables describing the current distribution of dislocations, should suffice to describe the state of a deformed material element. The resulting general theory was shown to be most efficiently expressed in a spatial (Eulerian) format reminiscent of the rheological models employed in fluid mechanics. The elimination of one finite deformation measure and a reformulation in terms of more easily 'visualized' Eulerian state variables were proclaimed as the principal advantages of this alternative approach.

Notwithstanding such claims, the traditional Lagrangian format admittedly benefits from a number of desirable features. Most apparent is that the differential evolution equations for Lagrangian inelastic state variables can, in general, be specified to within an inelastic rate term, with vanishing rate $(\dot{\mathbf{A}} = \mathbf{0})$ appropriate for elastic deformation. The alternative format presented in (Dashner, 1986b) suffers by comparison in that each Eulerian state variable must first be explicitly defined before the elastic and inelastic terms in its rate law can be identified. For example, a specific model incorporating a concept of "dislocation strain" was assumed to depend on a metric-type tensor $\mathbf{c}^*$ which was shown to evolve according to the rate law

$$\overset{\circ}{\mathbf{c}}{}^* = \mathbf{c}^*(\mathbf{d}^* - \mathbf{D}) + (\mathbf{d}^* - \mathbf{D})\mathbf{c}^*$$

expressed in terms of the Jaumann (corotational) derivative $\overset{\circ}{(\;)}$, the rate of deformation tensor $\mathbf{D}$, and a non-elastic, symmetric deformation rate tensor $\mathbf{d}^*$.

This complication gives rise to yet another. Recall that, in the traditional format, the stress response expression

$$\mathbf{S} = \rho_o \frac{\partial \psi}{\partial \mathbf{E}}$$

[1] through the formalized notion of invariance under "element replacement"

involving the symmetric Piola-Kirchhoff stress $\mathbf{S}$, the free energy ψ, and the total Green strain $\mathbf{E}$, is not sensitive to the specification of additional state variables. In contrast to this, it appears that the stress response equation in the alternative format cannot be obtained until all state variables are defined and their elastic evolution laws determined. For the dislocation strain model cited above, stress response was ultimately shown to be determined by the expression

$$\boldsymbol{\sigma} = -2\rho\, sym\left[\left(\frac{\partial\psi}{\partial\mathbf{c}}\mathbf{c}\right) + \left(\frac{\partial\psi}{\partial\mathbf{c}^*}\mathbf{c}^*\right)\right]$$

expressed in terms of the metric $\mathbf{c}^*$, the Cauchy stress $\boldsymbol{\sigma}$, and an elastic deformation tensor $\mathbf{c}$. For these reasons, general theoretical developments such as those of Hill (1959, 1968), and Hill *et al.* (1973), concerning uniqueness and stability might initially appear to be accessible only within the classical format.

Apart from the difficulty arising from an inappropriate invariance requirement, one could reasonably contend that these desirable features justify the introduction of an extra geometric 'dummy variable.' However, since few seem to appreciate the physical insignificance of the accumulated plastic strain, or are aware of the mathematical restrictions which should limit its role in constitutive equations, the real cost should be measured in the expenditure of time and resources on physically unrealistic models. Fortunately, this compromise can be avoided as the Eulerian state variable format introduced in (Dashner, 1986b) can be recast in a new form in which most of the advantages of the traditional format are retained - without introducing any physically irrelevant variables.

Section 2 presents a synopsis of the alternative constitutive forms developed in (Dashner, 1986b). This begins with the specification of the deformation measures needed to adequately describe the current state of elastic deformation, and a brief discussion of the potential role of additional "dislocation related" state variables. Next, the state variable dependent *response* and *evolution* functions are presented in conjunction with the rotational invariance requirements necessary to insure frame invariance, and to properly account for any symmetry of the underlying cell structure. In addition, kinematical considerations lead to the identification of the purely elastic and inelastic terms in the rate equations for each of the elastic deformation variables.

Section 3 introduces the differential (local) elastic *stretching* and *unstretching* maps which are then used to define a material element's instantaneous 'shadow' configuration and, over time, its 'shadow flow.' Examination of the kinematics of this secondary *shadow flow* reveals the significance of its flow rate (velocity gradient) tensor, and that of its symmetric *stretching* and antisymmetric *spin* components. Of particular interest is that the shadow stretching (rate of deformation) tensor is a natural measure of the plastic deformation rate, and that the spin of the shadow flow relative to the corotating frame is determined (using a solution of Scheidler, 1994) from the tensor sum of the plastic and actual deformation rates. This explicit solution for the shadow flow spin is then used to define a 'shadow frame' and the 'shadow time derivative' respectively, as a corotating frame, and corotational time derivative for this shadow flow. This makes it possible to reformulate the evolution equations for the elastic variables in terms of their shadow rates, and a pair of evolution functions which determine the symmetric shadow stretching (rate of plastic deformation), and an antisymmetric 'plastic spin' tensor. This last tensor fixes the rotation (relative to the shadow frame) of the characteristic reference cell within the elastically unstretched shadow flow.

Inspired by the recent work of Xiao *et al.* (1997), Section 4 assembles the mathematical results needed to replace dependence on the symmetric elastic stretch tensor with its natural log, that is, the *natural elastic strain*. It is noteworthy that Xiao's analysis showed that there exists a special (rotating) reference frame relative to which the time derivative of the total natural strain is exactly equal to the rate of deformation tensor for the material flow. Here, it is shown that there exists a special reference frame relative to which the time derivative of the elastic natural strain is exactly equal to the difference of the deformation rate tensors for the material and shadow flows - the latter being equal to the rate of plastic deformation. Unfortunately, this special frame is neither the corotating nor shadow frames and, consequently, does not appear to be easily characterized or of much immediate utility. This section concludes with a derivation of the appropriate evolution equations for the natural elastic strain in terms of both its corotational and shadow time rate.

Section 5 defines a specific group transformation operator which is then used to replace the original set of Eulerian dislocation state variables with a related 'semi-Lagrangian' set. It is of critical importance that each of these new variables is perceived by shadow frame observers to remain constant during any purely elastic deformation process. This 'change of variables' allows for a theoretical reformulation expressed entirely in terms of shadow rates and *inelastic* evolution functions, that is, functions which vanish during purely elastic deformation processes. This section concludes with a complete restatement of the general theory in three distinct forms. While all involve the same unspecified collection of (dislocation related) semi-Lagrangian inelastic state variables, the first employs a description of elastic deformation in terms of a single gradient-type 'cell placement' tensor, while the second relies instead upon specification of symmetric elastic stretch and orthogonal cell orientation tensors. The third featured form results from the replacement of the symmetric elastic stretch with the aforementioned finite elastic (natural) log-strain tensor. Perhaps not surprisingly, these log-strain forms are distinguished by their relative compactness, and later prove to be particularly revealing under a variety of special circumstances.

Section 6 builds the mathematical foundation for the thermodynamic considerations to follow. Recognizing the importance of purely elastic deformation, it begins with the introduction of an additive decomposition of the shadow rate time derivative of a tensor-valued state function into *elastic* and *non-elastic* components - the former being the shadow rate with all inelastic evolution functions set to zero. After applying this decomposition to all existing rate forms, a general expression for the time derivative of a scalar-valued function of state is developed. This is in anticipation of its subsequent application to the internal energy and yield functions.

Within the purely mechanical context of this paper, Section 7 explores the theoretical implications of the requirement of conformity with the second law of thermodynamics. This multifold discussion embraces a wide range of materials which may, or may not, be *non-viscous*, *rate independent*, *elastically compliant*, *structurally isotropic*, have *invariant elastic properties*, or *purely dissipative inelastic mechanisms*. In all cases, specific forms for Kirchhoff stress response and the rate of mechanical dissipation per unit of reference volume are deduced. The implications of *small elastic strains* are also assessed.

Section 8 provides mathematical support for the (closed deformation cycle) material stability considerations of the concluding section. Here, a secondary set of state variables is assigned to each state during a continuing deformation process initiated from some specified base state. After developing a list of their useful properties, it is shown that these so-called 'elastic recovery variables' are, in fact, the actual state (variable) values that would be measured by shadow frame observers if the current deformation process were to be elastically 'closed', that is, completed through a purely elastic deformation process which recovers the base state material geometry. This section concludes with the derivation of rate equations for these *recovery variables*, and for the shadow rates of functions which depend on them.

As a final exercise, Section 9 presents a derivation of the constitutive inequalities associated with the requirement of material stability in the sense of Il'iushin (1961) for a special class of rate independent solids. Of particular interest is the demonstration that these reduce to the familiar yield surface normality and convexity conditions of Drucker (1951) for the class of non-viscous, rate independent, elastically compliant, structurally isotropic solids having invariant elastic properties. It is noteworthy that this is proved for material models involving any number of dislocation state variables, subject to arbitrarily large elastic and/or plastic strains.

In summary, the intent of this work is to elucidate a new theoretical structure for the modeling of both crystalline and polycrystalline solids. The general theory, as presented, is fully frame invariant, explicitly accounts for finite deformation, arbitrarily large elastic and plastic strains, persistent anisotropic elastic cell structure (structural anisotropy), and possesses the capability (through the introduction of state variables) of modeling the myriad of induced isotropic and anisotropic effects associated with the evolution of local dislocation structure. The inadequacy of the classical theory, arising from its restrictive mathematical form and the limited physical relevance of the total and plastic Lagrangian strain tensors, provides the impetus for this effort. By comparison, it is claimed that this new theoretical structure is more general than the classical Green-Naghdi forms which were evidently based on the presumption that the variables $\mathbf{E}, \mathbf{E}_p$ and a collection of 'dislocation-driven' state variables, in addition to a collection of fixed 'structure tensors,' would provide a platform sufficient for the modeling of anisotropic inelastic solids. The addition of a new structural orientation or "corotation" tensor variable, as recently envisaged by Casey (1987), would restore the required level of generality at the expense of economy and physical clarity. This insofar as it does not explicitly recognize the physically motivated restriction of invariance under 'element replacement' as postulated in (Dashner, 1986a).

2. Theoretical structure

In contrast to the classical referential or Lagrangian theory of Green and Naghdi (Green *et al.*, 1965), the general theoretical structure set forth in (Dashner, 1986b) is of the Eulerian type. It is based on the assumption that the 'state' of a deformed differential element of crystalline or polycrystalline solid-like material in its *current* configuration is fixed by the instantaneous 'placement' of a characteristic elastic reference cell, and an 'adequate' description of the instantaneous dislocation distribution in terms of a finite collection of Eulerian tensor state variables. Once an 'undislocated' (virgin) cubic reference cell has been set aside, the relevant state variables identified, and the *response* and *evolution* functionals endowed with certain smoothness properties, this general inelastic theory {cf. Dashner (1986b), (3.28, 30, 32)} takes the form

$$\begin{aligned} \boldsymbol{R} &= [\sigma,\psi] = \Re(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}), \\ \overset{\circ}{\mathbf{F}}_e &= \dot{\mathbf{F}}_e - \mathbf{W}\mathbf{F}_e = \mu_e(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}), \\ \overset{\circ}{\mathbf{q}}_\alpha &= \mu_\alpha(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) \ ; \quad \alpha = 1, \ldots, \mathrm{N}. \end{aligned} \tag{2.1}$$

These relations determine the instantaneous element *response*, Cauchy stress and free energy, in terms of a *response function* $\Re$ which is dependent on the *cell placement* tensor $\mathbf{F}_e$, the aforementioned collection of *dislocation state variables* $\boldsymbol{q} \equiv \{\mathbf{q}_\alpha\}_{\alpha=1}^{\mathrm{N}}$, and a finite set $\boldsymbol{A} \equiv \{\mathbf{A}_k\}_{k=1}^{\mathrm{M}}$ of *Rivlin-Ericksen* tensors

$$\begin{aligned} \mathbf{A}_\emptyset &\equiv \mathbf{I} \ \text{(identity tensor)}, \\ \mathbf{A}_1 &\equiv 2\mathbf{D}, \\ \mathbf{A}_{k+1} &\equiv \dot{\mathbf{A}}_k + \mathbf{A}_k\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_k \ ; \quad k = 1, 2, \ldots . \end{aligned} \tag{2.2}$$

These relations are expressed in terms of the material flow *velocity gradient* $\mathbf{L} = \nabla\vec{v}$ and its associated *rate of deformation* (stretching) and *flow vorticity* (spin) components

$$\begin{aligned} \mathbf{D} &= sym(\mathbf{L}) = [\mathbf{L}]_S \equiv \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}), \\ \mathbf{W} &= asym(\mathbf{L}) = [\mathbf{L}]_A \equiv \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}). \end{aligned} \tag{2.3}$$

Variation of these state descriptors over time is determined by a set of *state variable evolution* functions μ_c & $\{\mu_\alpha\}_{\alpha=1}^{N}$. The notation $\overset{o}{\overline{(\ )}}$ is used here and throughout this manuscript to denote the frame invariant *corotational* or Jaumann time derivative[2].

It is significant that these constitutive relations show no explicit dependence on the *total deformation* of the material element relative to some preselected undeformed reference configuration[3]. It is this fundamental characteristic that distinguishes the present theoretical structure from the classical one of Green and Naghdi. It is my belief and contention that this approach to inelastic modeling is consistent with the thinking of a growing number of researchers whose work is based on a conceptual understanding of inelastic mechanisms owing to the foundational contributions of Mandel (1971).

The axiom of *material frame invariance* makes it necessary {cf. Dashner (1986b), $(3.29, 31, 33, 34)_1$} to require that these *response* and *evolution* functions satisfy the relations

$$\begin{aligned} \mathbf{T}_Q[\mathfrak{R}(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A})] &= \mathfrak{R}(\mathbf{T}_Q\mathbf{F}_e, \mathbf{T}_Q\boldsymbol{q}, \mathbf{T}_Q\boldsymbol{A}), \\ \mathbf{Q}[\mu_e(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A})] &= \mu_e(\mathbf{T}_Q\mathbf{F}_e, \mathbf{T}_Q\boldsymbol{q}, \mathbf{T}_Q\boldsymbol{A}), \\ \mathbf{T}_Q[\mu_\alpha(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A})] &= \mu_\alpha(\mathbf{T}_Q\mathbf{F}_e, \mathbf{T}_Q\boldsymbol{q}, \mathbf{T}_Q\boldsymbol{A}) \;;\quad \alpha = 1, \dots, N, \end{aligned} \tag{2.4}$$

for each $\mathbf{Q}$ from the full proper orthogonal group Θ. Here, as in {Dashner (1986b), (3.4)}, $\mathbf{T}_Q$ is used to represent the appropriate tensor transformation operator associated with the proper orthogonal element (post) rotation tensor $\mathbf{Q}$, *e.g.*

$$\begin{aligned} \boldsymbol{R} = [\sigma, \psi] &\rightarrow \mathbf{T}_Q\boldsymbol{R} = [\mathbf{Q}\sigma\mathbf{Q}^T, \psi], \\ \mathbf{F}_e &\rightarrow \mathbf{T}_Q\mathbf{F}_e = \mathbf{Q}\mathbf{F}_e, \\ \mathbf{q}_\alpha &\rightarrow {}_\alpha\mathbf{T}_Q\mathbf{q}_\alpha \;;\quad \alpha = 1, \dots, N, \\ \boldsymbol{q} = \{\mathbf{q}_\alpha\}_{\alpha=1}^{N} &\rightarrow \mathbf{T}_Q\boldsymbol{q} = \{{}_\alpha\mathbf{T}_Q\mathbf{q}_\alpha\}_{\alpha=1}^{N}, \\ \boldsymbol{A} = \{\mathbf{A}_k\}_{k=1}^{M} &\rightarrow \mathbf{T}_Q\boldsymbol{A} = \{\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T\}_{k=1}^{M}. \end{aligned} \tag{2.5}$$

The restrictions associated with material symmetry {cf. Dashner (1986b), $(3.29, 31, 33, 34)_2$} are stated in terms of the orthogonal symmetry group $\wp$ for the selected reference cell. Invariance under certain pre-rotations of this characteristic cell is properly enforced by requiring that

---

[2] Recognizing that $(2.1)_2$ is not consistent with the "standard form" of the *corotational* derivative of a second order tensor, it is worthwhile to recall that $\mathbf{F}_e$ is not a second order tensor of the same type as, say, the Cauchy stress σ, which transforms according to the rule

$$\sigma \rightarrow \mathbf{Q}\sigma\mathbf{Q}^T$$

under a superposed rigid rotation $\mathbf{Q}$ of the deformed material element. Indeed, by definition $\mathbf{F}_e$ is what is classically referred to as a "two-point" tensor having its "right foot" (dyadically speaking) firmly planted in the preselected fixed reference cell, and its "left foot" in the deformed material element. Because of this, it transforms according to the rule

$$\mathbf{F}_e \rightarrow \mathbf{Q}\mathbf{F}_e$$

under a superposed rigid rotation. This is fully developed in Sec 3.A of (Dashner, 1986b).

[3] It is certainly possible to reintroduce such dependence by identifying one of the state variables as a total deformation measure. Relying on the arguments presented in (Dashner, 1986a), it is this author's *conjecture* that there exist no real materials for which such an identification will be appropriate.

$$\begin{aligned}&\Re(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) = \Re(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{q}, \boldsymbol{A}),\\ &\mu_e(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) = [\mu_e(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{q}, \boldsymbol{A})]\mathbf{Q},\\ &\mu_\alpha(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) = \mu_\alpha(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{q}, \boldsymbol{A}) \ ; \quad \alpha = 1, \dots, N,\end{aligned} \tag{2.6}$$ [4]

for each $\mathbf{Q} \in \wp$.

A particularly convenient (equivalent) form for the evolution equation $(2.1)_2$ {cf. Dashner (1986b), $(3.40)_1$, 3.42)} is

$$\dot{\mathbf{F}}_e = \mathbf{L}\mathbf{F}_e - \mathbf{F}_e\mathcal{L}_p, \tag{2.7}$$ [5]

expressed in terms of the inelastic (plastic) velocity gradient[6] function

$$\mathcal{L}_p = \hat{\mathcal{L}}_p(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) = \mathbf{F}_e^{-1}\{\tfrac{1}{2}\mathbf{A}_1\mathbf{F}_e - \mu_e(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A})\} \ ; \quad [\mathbf{A}_1 = 2\mathbf{D}],$$

subject to the invariance requirements

$$\hat{\mathcal{L}}_p(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) = \begin{cases} \hat{\mathcal{L}}_p(\mathbf{Q}\mathbf{F}_e, \mathbf{T}_Q\boldsymbol{q}, \mathbf{T}_Q\boldsymbol{A}) \ ; & \text{for each } \mathbf{Q} \in \Theta, \\ \mathbf{Q}^T[\hat{\mathcal{L}}_p(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{q}, \boldsymbol{A})]\mathbf{Q} \ ; & \text{for each } \mathbf{Q} \in \wp. \end{cases} \tag{2.8}$$

With reference to {Dashner (1986b), (3.45)}, this inelastic rate term, which vanishes during any purely elastic deformation process, "is seen to represent the velocity gradient associated with the material flow in (through) the reference cell." This flow rate tensor can, of course, be decomposed into its symmetric *stretching* and antisymmetric *spin* components

$$\mathcal{D}_p \equiv [\mathcal{L}_p]_S = \tfrac{1}{2}(\mathcal{L}_p + \mathcal{L}_p^T) \quad \& \quad \mathcal{W}_p \equiv [\mathcal{L}_p]_A = \tfrac{1}{2}(\mathcal{L}_p - \mathcal{L}_p^T),$$

each of which must be specified in accordance with the same invariance criteria (2.8). As evidenced by the example of the theory of dislocation strain {cf. Dashner (1986b), Sec. 4B}, a similar decomposition of the μ_α evolution functions into purely elastic and inelastic parts is an essential step in the construction of a specific theory. Unfortunately, it appears that each state variable would have to be explicitly defined before this could be accomplished. As emphasized in the introduction, this stands as an impediment to the type of classical (thermodynamic) theoretical development generally associated with Lagrangian-type formulations. A principal objective of this communication is to eliminate this comparative disadvantage.

Kinematically, elastic reference cell placement in the current configuration can be regarded as a two step process through the polar decomposition

$$\mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e, \tag{2.9}$$

[4] These forms are equivalent to those cited from (Dashner, 1986b) although they differ in appearance due to the replacement of $\mathbf{Q} \in \wp$ with $\mathbf{Q}^T \in \wp$. The present forms are thought to be more consistent with established convention.

[5] While this is fully developed in the cited reference, it is a simple matter to verify that the rate expression

$$\dot{\mathbf{F}}_e \doteq \mathbf{L}\mathbf{F}_e$$

holds during any *purely elastic* deformation process. This follows from the fact that bond structure is *materially embedded* (*i.e.* dragged along by the flow) during any such process, ensuring that

$$\mathbf{F}_e(t) \doteq \mathbf{F}(t)\mathbf{F}_{e_0}$$

during any *purely elastic* deformation process $\{\mathbf{F}(t) : \mathbf{F}(0) = \mathbf{I}\}$ imposed on a reference state in which $\mathbf{F}_e(0) = \mathbf{F}_{e_0}$.

[6] The symbol $\mathcal{L}_p$ is used here in place of $\boldsymbol{\Lambda}$ from (Dashner, 1986b).

where the *elastic* (left) *stretch* and *cell orientation* tensors are expressed as

$$\mathbf{V}_e = \mathbf{b}^{1/2} \ ; \quad \mathbf{R}_e = \mathbf{c}^{1/2}\mathbf{F}_e \tag{2.10}$$

in terms of the Eulerian elastic deformation tensors[7]

$$\mathbf{b} \equiv \mathbf{F}_e\mathbf{F}_e^{\mathrm{T}} = \mathbf{V}_e^2 \quad \& \quad \mathbf{c} \equiv \mathbf{b}^{-1} = \mathbf{F}_e^{-\mathrm{T}}\mathbf{F}_e^{-1} . \tag{2.11}$$

This makes it possible to interpret the cell placement mapping

$$\vec{Z} \equiv \mathbf{F}_e\vec{X}$$

as a simple rotation of the reference cell elements (directors) $\vec{X}$

$$\vec{Y} \equiv \mathbf{R}_e\vec{X}, \tag{2.12}$$

followed by a pure stretching deformation

$$\vec{Z} = \mathbf{V}_e\vec{Y} \ ; \quad \mathbf{V}_e = \mathbf{b}^{1/2},$$

which determines the final placement $\vec{Z}$ of a lattice cell director $\vec{X}$ in the current configuration. It is noteworthy that, for any orthogonal $\mathbf{Q}\in\Theta$, these elastic deformation measures satisfy the transformation relations

$$\left.\begin{array}{lll} \mathbf{V}_e & \to & \mathbf{T}_Q\mathbf{V}_e = \mathbf{Q}\mathbf{V}_e\mathbf{Q}^{\mathrm{T}} \\ \mathbf{R}_e & \to & \mathbf{T}_Q\mathbf{R}_e = \mathbf{Q}\mathbf{R}_e \end{array}\right\} \Rightarrow \mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e \to \mathbf{T}_Q\mathbf{F}_e = \mathbf{Q}\mathbf{F}_e ,$$
$$\mathbf{b} \to \mathbf{T}_Q\mathbf{b} = \mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}} ,$$
$$\mathbf{c} \to \mathbf{T}_Q\mathbf{c} = \mathbf{Q}\mathbf{c}\mathbf{Q}^{\mathrm{T}} ,$$

under a post-rotation of the deformed material element, and (2.13)

$$\left.\begin{array}{lll} \mathbf{V}_e & \to & \mathbf{P}_Q\mathbf{V}_e = \mathbf{V}_e \\ \mathbf{R}_e & \to & \mathbf{P}_Q\mathbf{R}_e = \mathbf{R}_e\mathbf{Q}^{\mathrm{T}} \end{array}\right\} \Rightarrow \mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e \to \mathbf{P}_Q\mathbf{F}_e = \mathbf{F}_e\mathbf{Q}^{\mathrm{T}} ,$$
$$\mathbf{b} \to \mathbf{P}_Q\mathbf{b} = \mathbf{b} ,$$
$$\mathbf{c} \to \mathbf{P}_Q\mathbf{c} = \mathbf{c} ,$$

under a pre-rotation of the virgin reference cell.

From this second set of relations, it is clear that the relevance of $\mathbf{R}_e$ as a state descriptor depends on the prominence of the reference cell's invariant directional characteristics. If this characteristic cell exhibits structural anisotropy, specification of $\mathbf{R}_e$ is clearly necessary to establish its proper orientation in the current element configuration. These considerations give rise to the invariance requirements (2.6) expressed in terms of the reference cell symmetry group $\wp$. However, if the elastic reference cell is structurally *isotropic* ($\wp = \Theta$), then it is equally clear that the elastic structure in the current configuration is independent of the specification of proper orthogonal $\mathbf{R}_e$. For this case, $\mathbf{R}_e$ may as well be (or would be indistinguishable from) the identity tensor $\mathbf{I}$, leaving a symmetric, positive definite placement tensor $\mathbf{F}_e = \mathbf{b}^{1/2}$, and an adequate description of the elastic constituent of state in terms of $\mathbf{b}$. With reference to

---

[7] respectively, the left and right elastic Cauchy-Green tensors

{Dashner (1986b), (3.35)}, full structural isotropy is seen to allow for a simplified constitutive representation of the form

$$\boldsymbol{R} = [\sigma, \psi] = \Re(\mathbf{e}, \boldsymbol{q}, \boldsymbol{A}),$$
$$\overset{\circ}{\mathbf{e}} = \dot{\mathbf{e}} + \mathbf{eW} - \mathbf{We} = \tilde{\mu}_e(\mathbf{e}, \boldsymbol{q}, \boldsymbol{A}),$$
$$\overset{\circ}{\mathbf{q}}_\alpha = \tilde{\mu}_\alpha(\mathbf{e}, \boldsymbol{q}, \boldsymbol{A}) \ ; \quad \alpha = 1, \ldots, N,$$

expressed in terms of any elastic strain tensor defined through an invertible, isotropic tensor mapping

$$\mathbf{e} = \tilde{\mathbf{e}}(\mathbf{b}).$$

As per {Dashner (1986b), (3.36)}, these functions are subject to the frame invariance constraint

$$\mathrm{T}_Q[f(\mathbf{e}, \boldsymbol{q}, \boldsymbol{A})] = f(\mathbf{QeQ}^T, \mathrm{T}_Q\boldsymbol{q}, \mathrm{T}_Q\boldsymbol{A}) \ ; \quad \text{for each } \mathbf{Q} \in \Theta.$$

Particular attention shall be focused on these reduced "structurally isotropic" forms as they are perceived to have the most immediate application. The particular choice of the Eulerian elastic (natural) log-strain tensor

$$\mathbf{e} = \mathbf{a} \equiv ln(\mathbf{V}_e) = ln(\mathbf{b}^{1/2}) = \tfrac{1}{2} ln(\mathbf{b})$$

shall also be shown to have interesting consequences.

Regardless of the degree of structural anisotropy of the reference cell, it is evident that the placement of the elastic stretch ellipsoid in the current configuration through the specification of $\mathbf{b}$, $\mathbf{c} = \mathbf{b}^{-1}$, or any related strain measure $\mathbf{e} = \tilde{\mathbf{e}}(\mathbf{b})$, is essential to the description of state. At this point, it is appropriate to recall {cf. Dashner (1986b), $(3.40)_{2,3}$} the evolution forms

$$\dot{\mathbf{b}} - \mathbf{bL}^T - \mathbf{Lb} = 2\boldsymbol{\Sigma}_b,$$
$$\dot{\mathbf{c}} + \mathbf{cL} + \mathbf{L}^T\mathbf{c} = 2\boldsymbol{\Sigma}_c,$$

expressed in terms of the symmetric inelastic "slippage" tensors[8] $\boldsymbol{\Sigma}_b$ and $\boldsymbol{\Sigma}_c$.

In view of the relations (2.7) and (2.11), and the identity

$$\mathbf{cb} = \mathbf{I} \quad \Rightarrow \quad \dot{\mathbf{c}} = -\mathbf{c}\dot{\mathbf{b}}\mathbf{c},$$

it is a simple exercise to show that

$$\dot{\mathbf{b}} - \mathbf{bL}^T - \mathbf{Lb} = -2\mathbf{b}^{1/2}\mathbf{D}_p\mathbf{b}^{1/2} \quad \Rightarrow \quad \boldsymbol{\Sigma}_b = -\mathbf{b}^{1/2}\mathbf{D}_p\mathbf{b}^{1/2},$$
$$\dot{\mathbf{c}} + \mathbf{cL} + \mathbf{L}^T\mathbf{c} = 2\mathbf{c}^{1/2}\mathbf{D}_p\mathbf{c}^{1/2} \quad \Rightarrow \quad \boldsymbol{\Sigma}_c = \mathbf{c}^{1/2}\mathbf{D}_p\mathbf{c}^{1/2},$$

$$\Rightarrow \left\{ \begin{array}{l} \overset{\circ}{\mathbf{b}} = \mathbf{bD} + \mathbf{Db} - 2\mathbf{b}^{1/2}\mathbf{D}_p\mathbf{b}^{1/2}, \\ \overset{\circ}{\mathbf{c}} = -\mathbf{cD} - \mathbf{Dc} + 2\mathbf{c}^{1/2}\mathbf{D}_p\mathbf{c}^{1/2}, \end{array} \right. \tag{2.14}$$

expressed in terms of a new *inelastic* (plastic) *deformation rate* (stretching) tensor

$$\mathbf{D}_p \equiv \mathbf{R}_e \mathcal{D}_p \mathbf{R}_e^T \ ; \quad \mathcal{D}_p \equiv [\mathcal{L}_p]_S = \tfrac{1}{2}(\mathcal{L}_p + \mathcal{L}_p^T). \tag{2.15}$$

---

[8] Here, the tensor symbol $\boldsymbol{\Sigma}$ replaces the symbol $\boldsymbol{\Gamma}$ from (Dashner, 1986b).

Based on the above interpretation of the inelastic flow tensor $\mathcal{L}_p$, it is clear that symmetric $\mathbf{D}_p$ can be physically interpreted as the rate of material deformation through the elastically unstretched, but rotated, reference cell. Moreover, in light of the rotational transforms (2.13) and invariance criteria (2.8) for the inelastic rate function $\hat{\mathcal{L}}_p$, it is clear that this new plastic deformation rate is determined by a constitutive function consistent with the Eulerian-type requirements

$$\hat{\mathbf{D}}_p(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) = \begin{cases} \mathbf{Q}^T[\hat{\mathbf{D}}_p(\mathbf{Q}\mathbf{F}_e, \mathbf{T}_Q\boldsymbol{q}, \mathbf{T}_Q\boldsymbol{A})]\mathbf{Q} \ ; & \text{for each } \mathbf{Q} \in \Theta, \\ \hat{\mathbf{D}}_p(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{q}, \boldsymbol{A}) \ ; & \text{for each } \mathbf{Q} \in \wp . \end{cases}$$

3. The elastic shadow

With reference to the cell placement decomposition relations (2.9) - (2.11), let $\vec{\lambda}$ represent a fixed *material* (as opposed to lattice) *director* in the current configuration, and $\vec{\boldsymbol{\lambda}}$ its correspondent defined through the local elastic *unstretching* and *stretching* maps

$$\vec{\boldsymbol{\lambda}} = \mathbf{u}\vec{\lambda} \quad \Leftrightarrow \quad \vec{\lambda} = \mathbf{v}\vec{\boldsymbol{\lambda}}, \tag{3.1}$$

expressed in terms of the positive definite, symmetric, Eulerian "stretch" and "unstretch" tensors

$$\begin{aligned} \mathbf{v} &\equiv \mathbf{V}_e = \mathbf{b}^{1/2} = \sqrt{\mathbf{F}_e\mathbf{F}_e^T} \quad \Rightarrow \quad \mathbf{F}_e = \mathbf{v}\mathbf{R}_e, \\ \mathbf{u} &\equiv \mathbf{v}^{-1} = \mathbf{c}^{1/2} = \mathbf{b}^{-1/2} = \mathbf{V}_e^{-1} \quad \Rightarrow \quad \mathbf{F}_e^{-1} = \mathbf{R}_e^T\mathbf{u}. \end{aligned} \tag{3.2}$$

These differential mapping are taken to define, at each instant, the so-called *unstretched* or *shadow* configuration relative to the current. Now, while each fixed material director $\vec{\lambda}$ evolves according to the rate law $\dot{\vec{\lambda}} = \mathbf{L}\vec{\lambda}$ during a continuing flow, the corresponding *shadow* director $\vec{\boldsymbol{\lambda}}$ is 'dragged-along' by the secondary 'shadow flow.' Thus, $\dot{\vec{\boldsymbol{\lambda}}} = \mathbf{L}_s\vec{\boldsymbol{\lambda}}$ in terms of the flow rate (velocity gradient) tensor $\mathbf{L}_s$ associated with this *shadow flow*. In terms of the actual and shadow flow rate tensors, differentiation of (3.1) yields

$$\begin{aligned} \dot{\vec{\lambda}} &= \dot{\mathbf{v}}\vec{\boldsymbol{\lambda}} + \mathbf{v}\dot{\vec{\boldsymbol{\lambda}}}, \\ \mathbf{L}\vec{\lambda} &= (\dot{\mathbf{v}} + \mathbf{v}\mathbf{L}_s)\vec{\boldsymbol{\lambda}}, \\ (\mathbf{L}\mathbf{v} - \dot{\mathbf{v}} - \mathbf{v}\mathbf{L}_s)\vec{\boldsymbol{\lambda}} &= \vec{0}, \end{aligned}$$

which implies that

$$\dot{\mathbf{v}} = \mathbf{L}\mathbf{v} - \mathbf{v}\mathbf{L}_s = \dot{\mathbf{v}}^T \quad \Rightarrow \quad \mathbf{L}\mathbf{v} - \mathbf{v}\mathbf{L}_s = \mathbf{v}\mathbf{L}^T - \mathbf{L}_s^T\mathbf{v},$$

$$\Rightarrow \quad \boxed{\dot{\mathbf{v}} = \begin{cases} \mathbf{L}\mathbf{v} - \mathbf{v}\mathbf{L}_s \\ \mathbf{v}\mathbf{L}^T - \mathbf{L}_s^T\mathbf{v} \end{cases}}.$$

Since $\dot{\mathbf{u}} = -\mathbf{u}\dot{\mathbf{v}}\mathbf{u}$, this further implies that (3.3)

$$\Rightarrow \quad \boxed{\dot{\mathbf{u}} = \begin{cases} \mathbf{L}_s\mathbf{u} - \mathbf{u}\mathbf{L} \\ \mathbf{u}\mathbf{L}_s^T - \mathbf{L}^T\mathbf{u} \end{cases}}.$$

After decomposing the shadow flow rate tensor into its *stretching* and *spin* components, *viz*.

$$\left.\begin{array}{l} \mathbf{D}_s \equiv [\mathbf{L}_s]_S = \frac{1}{2}(\mathbf{L}_s + \mathbf{L}_s^T) \\ \mathbf{W}_s \equiv [\mathbf{L}_s]_A = \frac{1}{2}(\mathbf{L}_s - \mathbf{L}_s^T) \\ \mathbf{\Omega}_s \equiv \mathbf{W}_s - \mathbf{W} \;\Rightarrow\; \mathbf{W}_s = \mathbf{\Omega}_s + \mathbf{W} \end{array}\right\} \Rightarrow \mathbf{L}_s = \mathbf{D}_s + \overbrace{(\mathbf{\Omega}_s + \mathbf{W})}^{\mathbf{W}_s}, \tag{3.4}$$

it is easily established that the corotational rates for the elastic *stretch* and *unstretch* tensors are given by

$$\boxed{\begin{array}{l} \overset{\circ}{\mathbf{v}} = \dot{\mathbf{v}} + \mathbf{vW} - \mathbf{Wv} = \left\{ \begin{array}{l} \mathbf{Dv} - \mathbf{v}(\mathbf{D}_s + \mathbf{\Omega}_s) \\ \mathbf{vD} - (\mathbf{D}_s - \mathbf{\Omega}_s)\mathbf{v} \end{array}\right. \\[2ex] \overset{\circ}{\mathbf{u}} = \dot{\mathbf{u}} + \mathbf{uW} - \mathbf{Wu} = \left\{ \begin{array}{l} (\mathbf{D}_s + \mathbf{\Omega}_s)\mathbf{u} - \mathbf{uD} \\ \mathbf{u}(\mathbf{D}_s - \mathbf{\Omega}_s) - \mathbf{Du} \end{array}\right. \end{array}}, \tag{3.5}$$

expressed in terms of the material and shadow flow deformation rates $\mathbf{D}$ and $\mathbf{D}_s$, and the increment of shadow flow spin, $\mathbf{\Omega}_s$, over and above that of the material flow.

Shadow flow stretching

Further interpretation of the shadow flow rate begins with $(3.2)_2$. In view of the above corotational rate relations, differentiation of the implied elastic deformation relation $\mathbf{c} = \mathbf{u}^2$ leads to

$$\begin{aligned} \overset{\circ}{\mathbf{c}} &= \overset{\circ}{\overline{\mathbf{u}^2}} \\ &= \mathbf{u}\overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{u}}\mathbf{u} \\ &= \mathbf{u}\{(\mathbf{D}_s + \mathbf{\Omega}_s)\mathbf{u} - \mathbf{uD}\} + \{\mathbf{u}(\mathbf{D}_s - \mathbf{\Omega}_s) - \mathbf{Du}\}\mathbf{u} \\ &= -\mathbf{u}^2\mathbf{D} - \mathbf{Du}^2 + 2\mathbf{uD}_s\mathbf{u} \end{aligned}$$

$$\overset{\circ}{\mathbf{c}} = -\mathbf{cD} - \mathbf{Dc} + 2\mathbf{uD}_s\mathbf{u},$$

and, since $\overset{\circ}{\mathbf{b}} = -\mathbf{b}\overset{\circ}{\mathbf{c}}\mathbf{b}$,

$$\overset{\circ}{\mathbf{b}} = \mathbf{bD} + \mathbf{Db} - 2\mathbf{vD}_s\mathbf{v}.$$

In view of $(3.2)_1$, direct comparison of this with $(2.14)_1$ leads to the immediate and important conclusion that

$$\boxed{\mathbf{D}_s = \mathbf{D}_p = \mathbf{R}_e\mathcal{D}_p\mathbf{R}_e^T} \;;\quad \mathbf{D}_s \equiv [\mathbf{L}_s]_S \quad \& \quad \mathcal{D}_p \equiv [\mathcal{L}_p]_S, \tag{3.6}$$

i.e., that the plastic deformation rate introduced in (2.15) is, in fact, just the stretching (rate of material deformation) tensor associated with this elastically unstretched shadow flow. This critical association shall henceforth be assumed with notational preference given to $\mathbf{D}_p$ over $\mathbf{D}_s$.

Shadow flow spin

In order to ascertain the significance of the *shadow flow spin* rate $\mathbf{\Omega}_s$, the *symmetric* and *antisymmetric product* fourth order tensors

$$\underset{\sim}{\mathbb{S}}_A\mathbf{X} \equiv \tfrac{1}{2}(\mathbf{AX} + \mathbf{XA}) \quad \& \quad \underset{\sim}{\mathbb{A}}_A\mathbf{X} \equiv \tfrac{1}{2}(\mathbf{AX} - \mathbf{XA}) \tag{3.7}$$

are brought into play. For any given symmetric tensor $\mathbf{A}$, these operators are clearly seen to linearly map second order tensors into second order tensors. Their algebraic properties, which have been studied in detail by a number of researchers, are reviewed in the Appendix. For present purposes, it is only necessary to know (and a simple matter to show) that $\underset{\sim}{\mathbf{S}}_A$ maps symmetric/antisymmetric tensors into symmetric/antisymmetric tensors, that $\underset{\sim}{\mathbf{A}}_A$ maps symmetric/antisymmetric tensors into antisymmetric/symmetric tensors, and that $\underset{\sim}{\mathbf{S}}_A$ is non-singular and therefore invertible whenever symmetric $\mathbf{A}$ is *definite*, either positive of negative. With specific reference to (A.3) and (A.14), these properties are concisely summarized by the expressions

$$\left[\underset{\sim}{\mathbf{S}}_A \cdot \mathbf{X}\right]^T = \underset{\sim}{\mathbf{S}}_A \cdot \mathbf{X}^T \quad \& \quad \left[\underset{\sim}{\mathbf{A}}_A \cdot \mathbf{X}\right]^T = -\left(\underset{\sim}{\mathbf{A}}_A \cdot \mathbf{X}^T\right),$$

for any $\mathbf{X}$ and $\mathbf{A} = \mathbf{A}^T$, and (3.8)

$$\mathbf{X} = \underset{\sim}{\mathbf{S}}_A \circ \underset{\sim}{\mathbf{S}}_A^{-1} \cdot \mathbf{X} = \underset{\sim}{\mathbf{S}}_A^{-1} \circ \underset{\sim}{\mathbf{S}}_A \cdot \mathbf{X},$$

for any $\mathbf{X}$ and <u>*definite*</u> $\mathbf{A} = \mathbf{A}^T$.

With this background, the desired relationship

$$\mathbf{u}\boldsymbol{\Omega}_s + \boldsymbol{\Omega}_s\mathbf{u} = \mathbf{u}(\mathbf{D} + \mathbf{D}_p) - (\mathbf{D} + \mathbf{D}_p)\mathbf{u}$$

is quickly seen to result from (3.6) and the dual relations $(3.5)_{3,4}$ for the corotational rate of the unstretch tensor $\mathbf{u}$. Expressed in terms of the above defined symmetric and antisymmetric product operators, this takes the form

$$\underset{\sim}{\mathbf{S}}_u \cdot \boldsymbol{\Omega}_s = \underset{\sim}{\mathbf{A}}_u \cdot (\mathbf{D} + \mathbf{D}_p),$$

leading to the unique solution

$$\boldsymbol{\Omega}_s = \underset{\sim}{\mathbf{S}}_u^{-1} \circ \underset{\sim}{\mathbf{A}}_u \cdot (\mathbf{D} + \mathbf{D}_p) = \left[\underset{\sim}{\mathbf{S}}_u^{-1} \circ \underset{\sim}{\mathbf{A}}_u\right] \cdot \mathbf{D} + \left[\underset{\sim}{\mathbf{S}}_u^{-1} \circ \underset{\sim}{\mathbf{A}}_u\right] \cdot \mathbf{D}_p. \tag{3.9}$$ [9]

In view of $(3.4)_3$, it is now clear that **the (extra) shadow flow spin $\boldsymbol{\Omega}_s$ is determined by the instantaneous value of the unstretch tensor $\mathbf{u}$, and the sum of the actual and plastic deformation rates $\mathbf{D}$ and $\mathbf{D}_s = \mathbf{D}_p$.**

In (A.34), the particular composition of operations employed above in the shadow spin rate solution is shown to be equivalent to the so-called *widget* operator as expressed by the identity

$$\underset{\sim}{\mathbf{W}}_A = \underset{\sim}{\mathbf{S}}_A^{-1} \circ \underset{\sim}{\mathbf{A}}_A. \tag{3.10}$$

This new operator is initially defined (for any definite $\mathbf{A} = \mathbf{A}^T$) in (A.32) through either of the equivalent expressions

$$\underset{\sim}{\mathbf{W}}_A \cdot \mathbf{X} \equiv \begin{cases} \mathbf{A}\left(\underset{\sim}{\mathbf{S}}_A^{-1} \cdot \mathbf{X}\right) - \mathbf{X}, \\ \mathbf{X} - \left(\underset{\sim}{\mathbf{S}}_A^{-1} \cdot \mathbf{X}\right)\mathbf{A}. \end{cases} \tag{3.11}$$

This important tensor operator plays a significant role in the developments to follow. With reference to (A.32) - (A.35), the relations

$$\begin{aligned} \mathbf{X} \bullet \left(\underset{\sim}{\mathbf{W}}_A \cdot \mathbf{Y}\right) &= \mathbf{Y} \bullet \left(\underset{\sim}{\mathbf{W}}_A \cdot \mathbf{X}\right), \\ \left[\underset{\sim}{\mathbf{W}}_A \cdot \mathbf{X}\right]^T &= -\left[\underset{\sim}{\mathbf{W}}_A \cdot \mathbf{X}^T\right], \\ \underset{\sim}{\mathbf{W}}_{A^{-1}} &= -\underset{\sim}{\mathbf{W}}_A, \end{aligned} \tag{3.12}$$

---

[9] An identical development using the dual relations $(3.5)_{1,2}$ leads to the alternative result

$$\boldsymbol{\Omega}_s = -\underset{\sim}{\mathbf{S}}_v^{-1} \circ \underset{\sim}{\mathbf{A}}_v \cdot (\mathbf{D} + \mathbf{D}_p).$$

summarize its essential properties as a symmetric bilinear form on the space of second order tensors which maps symmetric/antisymmetric tensors into antisymmetric/symmetric tensors. The third identity is immediately useful in that it, together with (3.10), facilitates a reformulation of the shadow spin expression (3.9) in either of the alternative forms

$$\boldsymbol{\Omega}_s = \begin{cases} \underset{\sim}{\mathbf{W}}_u \cdot (\mathbf{D}+\mathbf{D}_p) = \underset{\sim}{\mathbf{S}}_u^{-1} \circ \underset{\sim}{\mathbf{A}}_u \cdot (\mathbf{D}+\mathbf{D}_p), \\ -\underset{\sim}{\mathbf{W}}_v \cdot (\mathbf{D}+\mathbf{D}_p) = -\underset{\sim}{\mathbf{S}}_v^{-1} \circ \underset{\sim}{\mathbf{A}}_v \cdot (\mathbf{D}+\mathbf{D}_p) \;; \quad [\mathbf{v}=\mathbf{u}^{-1}]. \end{cases} \tag{3.13}$$

An explicit form for this shadow spin is also available by utilizing one of the many results established by Scheidler (1994). In this comprehensive work, it is shown that for definite $\mathbf{A} = \mathbf{A}^T$, the tensor equation

$$\underset{\sim}{\mathbf{S}}_A \cdot \mathbf{C} = \mathbf{X}$$

has the unique solution

$$\mathbf{C} = \underset{\sim}{\mathbf{S}}_A^{-1} \cdot \mathbf{X} = \{\mathbf{X} + \mathrm{III}_{^\dagger\mathbf{A}}^{-1} \cdot [{}^\dagger\mathbf{A}(\mathbf{XA} - \mathbf{AX})\,{}^\dagger\mathbf{A}]\}\mathbf{A}^{-1},$$

expressed in terms of (3.14)

$$\begin{aligned} \mathrm{I}_A &\equiv tr(\mathbf{A}) = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3, \\ \mathrm{II}_A &\equiv \tfrac{1}{2}[\mathrm{I}_A^2 - tr^2(\mathbf{A})] = \mathrm{A}_2\mathrm{A}_3 + \mathrm{A}_3\mathrm{A}_1 + \mathrm{A}_1\mathrm{A}_2, \\ \mathrm{III}_A &\equiv det(\mathbf{A}) = \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3, \\ {}^\dagger\mathbf{A} &\equiv \mathrm{I}_A\mathbf{I} - \mathbf{A}, \\ \mathrm{III}_{^\dagger\mathbf{A}} &\equiv det({}^\dagger\mathbf{A}) = \mathrm{I}_A\mathrm{II}_A - \mathrm{III}_A = (\mathrm{A}_2+\mathrm{A}_3)(\mathrm{A}_3+\mathrm{A}_1)(\mathrm{A}_1+\mathrm{A}_2), \end{aligned}$$

in which $\{\mathrm{A}_k\}_{k=1}^3$ are the principal values of $\mathbf{A}$[10]. With the aid of the fourth order $\mathbf{A}$-*bracket* operator

$$\underset{\sim}{\mathbf{B}}_A \cdot \mathbf{X} \equiv \mathbf{AXA} \;; \quad \begin{cases} [\underset{\sim}{\mathbf{B}}_A \cdot \mathbf{X}]^T = \underset{\sim}{\mathbf{B}}_A \cdot \mathbf{X}^T, \\ \mathbf{X} \bullet (\underset{\sim}{\mathbf{B}}_A \cdot \mathbf{Y}) = \mathbf{Y} \bullet (\underset{\sim}{\mathbf{B}}_A \cdot \mathbf{X}), \\ \underset{\sim}{\mathbf{B}}_A^{-1} = \underset{\sim}{\mathbf{B}}_{A^{-1}} \;; \quad [\mathbf{A} \sim \text{nonsingular}], \end{cases} \tag{3.15}$$

whose algebraic properties are also fully developed in the Appendix [(A.17) - (A.19)], this solution can be reexpressed in the compact form

$$\mathbf{C} = \underset{\sim}{\mathbf{S}}_A^{-1} \cdot \mathbf{X} = \{\mathbf{X} - 2\mathrm{III}_{^\dagger\mathbf{A}}^{-1} \cdot [\underset{\sim}{\mathbf{B}}_{^\dagger\mathbf{A}} \circ \underset{\sim}{\mathbf{A}}_A \cdot \mathbf{X}]\}\mathbf{A}^{-1}.$$

With this, and the defining expression $(3.11)_2$, it easily follows that

$$\begin{aligned} \underset{\sim}{\mathbf{W}}_A \cdot \mathbf{X} &= \mathbf{X} - (\underset{\sim}{\mathbf{S}}_A^{-1} \cdot \mathbf{X})\mathbf{A} \\ &= \mathbf{X} - \{[\mathbf{X} - 2\mathrm{III}_{^\dagger\mathbf{A}}^{-1} \cdot (\underset{\sim}{\mathbf{B}}_{^\dagger\mathbf{A}} \circ \underset{\sim}{\mathbf{A}}_A \cdot \mathbf{X})]\mathbf{A}^{-1}\}\mathbf{A} \\ \underset{\sim}{\mathbf{W}}_A \cdot \mathbf{X} &= 2\mathrm{III}_{^\dagger\mathbf{A}}^{-1} \cdot (\underset{\sim}{\mathbf{B}}_{^\dagger\mathbf{A}} \circ \underset{\sim}{\mathbf{A}}_A \cdot \mathbf{X}), \end{aligned}$$

which serves as proof of the additional *widget* relation

$$\underset{\sim}{\mathbf{W}}_A = 2\mathrm{III}_{^\dagger\mathbf{A}}^{-1} \cdot (\underset{\sim}{\mathbf{B}}_{^\dagger\mathbf{A}} \circ \underset{\sim}{\mathbf{A}}_A). \tag{3.16}$$

---

[10] While this solution is actually verified under more general circumstances {cf. Scheidler (1994), p. 137, eq. (5.6)}, this specialization to symmetric, definite $\mathbf{A}$ is entirely adequate for present purposes.

With the relations $(3.12)_3$, (3.14) and (3.15), this results in the explicit closed form expression

$$\underset{\sim}{\mathbb{W}}_A \mathbf{X} = -\left(\underset{\sim}{\mathbb{W}}_{A^{-1}} \mathbf{X}\right) = \frac{(\mathrm{I}_A\mathbf{I}-\mathbf{A})(\mathbf{AX}-\mathbf{XA})(\mathrm{I}_A\mathbf{I}-\mathbf{A})}{(\mathrm{I}_A\mathrm{II}_A-\mathrm{III}_A)} \tag{3.17}$$

for the evaluation of the *widget*[11]. Application of this to (3.13) then results in the explicit solutions

$$\boldsymbol{\Omega}_s = \left\{ \begin{array}{l} \underset{\sim}{\mathbb{W}}_u \mathbf{X} = \dfrac{(\mathrm{I}_u\mathbf{I}-\mathbf{u})(\mathbf{uX}-\mathbf{Xu})(\mathrm{I}_u\mathbf{I}-\mathbf{u})}{(\mathrm{I}_u\mathrm{II}_u-\mathrm{III}_u)} \\ -\underset{\sim}{\mathbb{W}}_v \mathbf{X} = \dfrac{(\mathrm{I}_v\mathbf{I}-\mathbf{v})(\mathbf{Xv}-\mathbf{vX})(\mathrm{I}_v\mathbf{I}-\mathbf{v})}{(\mathrm{I}_v\mathrm{II}_v-\mathrm{III}_v)} \end{array} \right\} ; \quad \mathbf{X} = \mathbf{D}+\mathbf{D}_p \tag{3.18}$$

for the *shadow flow spin.*

Yet another alternative representation involves the eigenvalues/vectors

$$\{\mathrm{a}_k, \hat{\mathrm{a}}_k\}_{k=1}^{3} \tag{3.19}$$

for the Eulerian elastic (natural) log strain tensor

$$\mathbf{a} \equiv ln(\mathbf{v}),$$

and the principal differences

$$\begin{aligned} \mu_1 &\equiv \mathrm{a}_2-\mathrm{a}_3 = ln(\mathrm{v}_2)-ln(\mathrm{v}_3) = ln(\mathrm{v}_2/\mathrm{v}_3), \\ \mu_2 &\equiv \mathrm{a}_3-\mathrm{a}_1 = ln(\mathrm{v}_3)-ln(\mathrm{v}_1) = ln(\mathrm{v}_3/\mathrm{v}_1), \\ \mu_3 &\equiv \mathrm{a}_1-\mathrm{a}_2 = ln(\mathrm{v}_1)-ln(\mathrm{v}_2) = ln(\mathrm{v}_1/\mathrm{v}_2). \end{aligned} \tag{3.20}$$

With reference to $(3.13)_2$ and $(A.42)_1$, this spin rate is seen to admit the alternative eigenbasis expansion

$$\begin{aligned} \boldsymbol{\Omega}_s &= -\underset{\sim}{\mathbb{W}}_v (\mathbf{D}+\mathbf{D}_p), \\ \boldsymbol{\Omega}_s &= tanh(\tfrac{1}{2}\mu_1)(\hat{\mathrm{a}}_2\bullet\mathbf{X}\hat{\mathrm{a}}_3)[(\hat{\mathrm{a}}_3\otimes\hat{\mathrm{a}}_2)-(\hat{\mathrm{a}}_2\otimes\hat{\mathrm{a}}_3)] \\ &\quad + tanh(\tfrac{1}{2}\mu_2)(\hat{\mathrm{a}}_3\bullet\mathbf{X}\hat{\mathrm{a}}_1)[(\hat{\mathrm{a}}_1\otimes\hat{\mathrm{a}}_3)-(\hat{\mathrm{a}}_3\otimes\hat{\mathrm{a}}_1)] \\ &\quad\quad + tanh(\tfrac{1}{2}\mu_3)(\hat{\mathrm{a}}_1\bullet\mathbf{X}\hat{\mathrm{a}}_2)[(\hat{\mathrm{a}}_2\otimes\hat{\mathrm{a}}_1)-(\hat{\mathrm{a}}_1\otimes\hat{\mathrm{a}}_2)] \; ; \quad \mathbf{X}=\mathbf{D}+\mathbf{D}_p . \end{aligned}$$

---

[11] It is a straightforward exercise to confirm the identity

$$\mathbf{A}[{}^{\dagger}\mathbf{A}(\mathbf{AX}-\mathbf{XA}){}^{\dagger}\mathbf{A}] + [{}^{\dagger}\mathbf{A}(\mathbf{AX}-\mathbf{XA}){}^{\dagger}\mathbf{A}]\mathbf{A} = (\mathrm{I}_A\mathrm{II}_A-\mathrm{III}_A)(\mathbf{AX}-\mathbf{XA})$$

through the application of the Cayley-Hamilton Theorem, *viz.*

$$\mathbf{A}^3-\mathrm{I}_A\mathbf{A}^2+\mathrm{II}_A\mathbf{A}-\mathrm{III}_A\mathbf{I}=\mathbf{0} \;\Rightarrow\; \mathbf{A}^3=\mathrm{I}_A\mathbf{A}^2-\mathrm{II}_A\mathbf{A}+\mathrm{III}_A\mathbf{I}.$$

After safely extracting this "rabbit from the hat," one need only observe that this identity can be reexpressed as

$$4\underset{\sim}{\mathbb{S}}_A \circ \underset{\sim}{\mathbb{B}}_{{}^{\dagger}A} \circ \underset{\sim}{\mathbb{A}}_A = 2\mathrm{III}_{{}^{\dagger}A}\underset{\sim}{\mathbb{A}}_A ,$$

which, for definite **A**, insures that (3.10) admits the alternative form

$$\underset{\sim}{\mathbb{W}}_A = \underset{\sim}{\mathbb{S}}_A^{-1} \circ \underset{\sim}{\mathbb{A}}_A = 2\mathrm{III}_{{}^{\dagger}A}^{-1}\underset{\sim}{\mathbb{B}}_{{}^{\dagger}A} \circ \underset{\sim}{\mathbb{A}}_A .$$

This constitutes a direct proof of (3.16) which, in conjunction with (3.11), provides an alternative proof for the above cited solution of Scheidler.

After recalling that every antisymmetric tensor $(\mathbf{Z} = -\mathbf{Z}^T)$ has a unique *axial vector*, *i.e.*

$$\vec{w} = \mathbf{Z}\vec{u} \quad \Leftrightarrow \quad \vec{w} = \vec{z} \times \vec{u},$$

it is then easily shown that the expression

$$\vec{\Omega}_s = tanh\left(\tfrac{1}{2}\mu_1\right)\left[\hat{a}_2 \bullet \mathbf{X}\hat{a}_3\right]\hat{a}_1 + tanh\left(\tfrac{1}{2}\mu_2\right)\left[\hat{a}_3 \bullet \mathbf{X}\hat{a}_1\right]\hat{a}_2 + tanh\left(\tfrac{1}{2}\mu_3\right)\left[\hat{a}_1 \bullet \mathbf{X}\hat{a}_2\right]\hat{a}_3 \ ; \quad \mathbf{X} = \mathbf{D} + \mathbf{D}_p \tag{3.21}$$

gives the axial (angular velocity) vector for the *shadow spin*.

Collecting the results (3.4), (3.6) and (3.13), the shadow flow rate tensor is expressed as

$$\mathbf{L}_s = \underbrace{\mathbf{D}_p}_{\mathbf{D}_s} + \underbrace{(\mathbf{\Omega}_s + \mathbf{W})}_{\mathbf{W}_s} \ ; \quad \left\{ \begin{array}{l} \mathbf{D}_p = \mathbf{R}_e \mathbf{\Phi}_p \mathbf{R}_e^T \\ \mathbf{\Omega}_s = \underset{\sim}{\mathbf{W}}_u (\mathbf{D} + \mathbf{D}_p) \\ \mathbf{W} = [\mathbf{L}]_A \end{array} \right. , \tag{3.22}$$

in terms of the rate of (plastic) stretching $\mathbf{D}_p = \mathbf{D}_s$ and the (extra) spin $\mathbf{\Omega}_s$ for the shadow flow.

Shadow frame and associated time rate

With this, it is now possible to define a local *shadow frame* as any frame of reference which adopts the rotational motion of a material elements shadow flow. Thus, a shadow frame is a corotating frame for this secondary flow. Analogous to the Jaumann or corotating time rate $\overset{o}{\overline{(\ )}}$, the *shadow rate*, $\overset{s}{\overline{(\ )}}$, is then defined as the time rate of change measured by a similarly oriented observer moving in a shadow frame. For a second order Eulerian tensor $\left[\mathbf{T}_Q\mathbf{B} = \mathbf{QBQ}^T\right]$, this shadow rate derivative takes the form

$$\overset{s}{\mathbf{B}} \equiv \dot{\mathbf{B}} + \mathbf{BW}_s - \mathbf{W}_s\mathbf{B} = \overset{o}{\mathbf{B}} + \mathbf{B\Omega}_s - \mathbf{\Omega}_s\mathbf{B} \ . \tag{3.23}$$

It is also useful to note that the non-invariant material and shadow flow rate tensors as observed from the *ground*, *corotating*, and *shadow frames* are given by

$$\begin{array}{lll} [\mathbf{L}]_{G.F.} = \mathbf{L} = \mathbf{D} + \mathbf{W} & ; \ [\mathbf{L}_s]_{G.F.} = \mathbf{L}_s = \mathbf{D}_p + \mathbf{W}_s & ; \ \mathbf{W}_s = \mathbf{\Omega}_s + \mathbf{W}, \\ [\mathbf{L}]_{C.F.} = \mathbf{D} & ; \ [\mathbf{L}_s]_{C.F.} = \mathbf{D}_p + \mathbf{\Omega}_s & \\ [\mathbf{L}]_{S.F.} = \mathbf{D} - \mathbf{\Omega}_s & ; \ [\mathbf{L}_s]_{S.F.} = \mathbf{D}_p & ; \ \mathbf{\Omega}_s = \left\{ \begin{array}{l} \underset{\sim}{\mathbf{W}}_u (\mathbf{D} + \mathbf{D}_p), \\ -\underset{\sim}{\mathbf{W}}_v (\mathbf{D} + \mathbf{D}_p). \end{array} \right. \end{array} \tag{3.24}$$

To illustrate the utility of these observations, note that the rate expressions (3.3) are seen to immediately imply the corotational rate relations (3.5), as well as the corresponding shadow rate relations

$$\overset{s}{\mathbf{v}} = \left\{ \begin{array}{l} (\mathbf{D} - \mathbf{\Omega}_s)\mathbf{v} - \mathbf{vD}_p \\ \mathbf{v}(\mathbf{D} + \mathbf{\Omega}_s) - \mathbf{D}_p\mathbf{v} \end{array} \right. \quad \& \quad \overset{s}{\mathbf{u}} = \left\{ \begin{array}{l} \mathbf{D}_p\mathbf{u} - \mathbf{u}(\mathbf{D} - \mathbf{\Omega}_s) \\ \mathbf{uD}_p - (\mathbf{D} + \mathbf{\Omega}_s)\mathbf{u} \end{array} \right. . \tag{3.25}$$

Reference cell spin

In the event of structural anisotropy, it will also be necessary (as previously noted) to fix the orientation of the elastically recovered reference cell in the shadow configuration through (2.12). In general, each observer will perceive reference cell rotation in the shadow configuration during any continuing deformation process. Thus, the rate of change of a fixed reference cell or 'lattice' director (as opposed to a fixed material director) in the shadow flow is given by

$$\dot{\vec{Y}} = \mathbf{W}_e \vec{Y}$$

expressed in terms of the *reference cell rotation rate* tensor

$$\mathbf{W}_e \equiv \dot{\mathbf{R}}_e \mathbf{R}_e^T = -\mathbf{W}_e^T .$$

After defining $\mathbf{\Omega}_e$ as the cell rotation rate relative to the shadow frame, it follows from $(3.4)_3$ that

$$\mathbf{\Omega}_e \equiv [\mathbf{W}_e]_{S.F.} = \mathbf{W}_e - \mathbf{W}_s = \mathbf{W}_e - \mathbf{\Omega}_s - \mathbf{W} ,$$
$$\Rightarrow \quad \mathbf{W}_e = \mathbf{\Omega}_e + \mathbf{\Omega}_s + \mathbf{W} ,$$

which, in view of the above relations (3.24), gives rise to the reference cell rotation rate expressions

$$\boxed{\begin{aligned} &\dot{\mathbf{R}}_e = \mathbf{W}_e \mathbf{R}_e \ ; \quad \{\mathbf{W}_e = \mathbf{\Omega}_e + \mathbf{\Omega}_s + \mathbf{W}\} \\ &\overset{o}{\mathbf{R}}_e = (\mathbf{\Omega}_e + \mathbf{\Omega}_s)\mathbf{R}_e \ ; \quad \{[\mathbf{W}]_{C.F.} = \mathbf{0}\} \\ &\overset{s}{\mathbf{R}}_e = \mathbf{\Omega}_e \mathbf{R}_e \ ; \quad \{[\mathbf{W}]_{S.F.} = -\mathbf{\Omega}_s\} \end{aligned}} \ . \tag{3.26}$$

Along similar lines, the rate forms (3.3) and $(3.26)_1$ allow for differentiation of the cell gradient decomposition $(3.2)_1$, resulting in the expression

$$\begin{aligned} \dot{\mathbf{F}}_e &= \dot{\mathbf{v}}\mathbf{R}_e + \mathbf{v}\dot{\mathbf{R}}_e \\ &= (\mathbf{L}\mathbf{v} - \mathbf{v}\mathbf{L}_s)\mathbf{R}_e + \mathbf{v}(\mathbf{W}_e \mathbf{R}_e) \\ &= \mathbf{L}\mathbf{F}_e - \mathbf{v}[(\mathbf{D}_p + \mathbf{\Omega}_s + \mathbf{W}) - (\mathbf{\Omega}_e + \mathbf{\Omega}_s + \mathbf{W})]\mathbf{R}_e \\ &= \mathbf{L}\mathbf{F}_e - (\mathbf{F}_e \mathbf{R}_e^T)(\mathbf{D}_p - \mathbf{\Omega}_e)\mathbf{R}_e \\ \dot{\mathbf{F}}_e &= \mathbf{L}\mathbf{F}_e - \mathbf{F}_e\left[\mathbf{R}_e^T(\mathbf{D}_p - \mathbf{\Omega}_e)\mathbf{R}_e\right] . \end{aligned}$$

Direct comparison of this with (2.7) then leads to the correspondence

$$\boxed{\mathcal{L}_p = \mathbf{R}_e^T(\mathbf{D}_p - \mathbf{\Omega}_e)\mathbf{R}_e \quad \Leftrightarrow \quad \left\{ \begin{aligned} &\mathbf{D}_p = \mathbf{R}_e \mathcal{D}_p \mathbf{R}_e^T \ ; \quad \mathcal{D}_p = [\mathcal{L}_p]_S \\ &\mathbf{\Omega}_e = -\mathbf{R}_e \mathcal{W}_p \mathbf{R}_e^T \ ; \quad \mathcal{W}_p = [\mathcal{L}_p]_A \end{aligned} \right.} \ , \tag{3.27}$$ [12]

between the inelastic rate measures $\mathcal{L}_p$, $\mathbf{D}_p$ and $\mathbf{\Omega}_e$, and to the frame specific rate relations

$$\boxed{\begin{aligned} &\dot{\mathbf{F}}_e = \mathbf{L}\mathbf{F}_e - \mathbf{F}_e \mathcal{L}_p \\ &\overset{o}{\mathbf{F}}_e = \mathbf{D}\mathbf{F}_e - \mathbf{F}_e \mathcal{L}_p \ ; \quad \{[\mathbf{L}]_{C.F.} = \mathbf{D}\} \\ &\overset{s}{\mathbf{F}}_e = (\mathbf{D} - \mathbf{\Omega}_s)\mathbf{F}_e - \mathbf{F}_e \mathcal{L}_p \ ; \quad \{[\mathbf{L}]_{S.F.} = \mathbf{D} - \mathbf{\Omega}_s\} \end{aligned}} \ . \tag{3.28}$$

[12] Note that the symmetric stretching relationship confirms the previously established result (3.6).

From (3.27) it immediately follows that, at any given state, specification of $\mathbf{D}_p$ and $\mathbf{\Omega}_c$ is equivalent to the specification of the reference cell material flow rate $\mathcal{L}_p$. For this reason, theoretical dependence on the inelastic evolution function (2.8) can (and will) be replaced through the specification of the new inelastic rate forms

$$\begin{aligned} \mathbf{D}_p &= \hat{\mathbf{D}}_p(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) \sim \{\text{plastic stretching as the rate of material deformation in } \textit{shadow flow}\}, \\ \mathbf{\Omega}_c &= \hat{\mathbf{\Omega}}_c(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) \sim \{\text{reference cell 'spin' in } \textit{shadow flow} \text{ relative to } \textit{shadow frame}\}. \end{aligned} \tag{3.29}$$

In light of the physical interpretation of $\mathcal{L}_p$[13], it is evident that both $\mathbf{D}_p$ and $\mathbf{\Omega}_c$ vanish during purely elastic deformations. Consequently, **during any *purely elastic* deformation process, a material element's secondary *shadow flow* is seen as nothing more than a *rigid rotation*** $[\mathbf{D}_s = \mathbf{D}_p = \mathbf{0}]$, **described** [through $(3.4)_3$ and (3.13)] **by the spin tensor**

$$\mathbf{W}_s = \mathbf{\Omega}_s + \mathbf{W} \; ; \quad \mathbf{\Omega}_s = \underset{\sim}{\mathbf{W}}_u \mathbf{D} = -\underset{\sim}{\mathbf{W}}_v \mathbf{D}.$$

Furthermore, [through $(3.26)_3$] **the characteristic *reference cell* is observed to remain fixed, or materially embedded, within this rigid *shadow flow*. Put differently, shadow frame observers perceive the shadow flow, with materially embedded reference cell, as absolutely *stationary* during purely elastic deformation processes.**

Subsequent developments shall depend upon specification of the evolution forms (3.29). It is therefore essential to note that the established invariance criteria (2.8) and (2.13), and the relations (3.27), make it necessary to subject these new inelastic rate functions to the Eulerian-type invariance restrictions

$$\begin{aligned} \mathsf{T}_Q[f(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A})] = \mathbf{Q}[f(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A})]\mathbf{Q}^T &= f(\mathbf{Q}\mathbf{F}_e, \mathsf{T}_Q\boldsymbol{q}, \mathsf{T}_Q\boldsymbol{A}) \; ; \quad \text{for each } \mathbf{Q} \in \Theta, \\ f(\mathbf{F}_e, \boldsymbol{q}, \boldsymbol{A}) &= f(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{q}, \boldsymbol{A}) \; ; \quad \text{for each } \mathbf{Q} \in \wp. \end{aligned} \tag{3.30}$$

Finally, it is readily confirmed {cf. Dashner (1986b), (4.36)} that the common assumption of *incompressible plastic flow* is embodied in the additional restrictions

$$\begin{aligned} \rho/\rho_o &= 1/det(\mathbf{F}_e) = 1/det(\mathbf{v}) = det(\mathbf{u}), \\ tr(\mathcal{L}_p) &= tr(\mathbf{L}_s) = tr(\mathbf{D}_p) = 0. \end{aligned} \tag{3.31}$$

4. Natural strain

It has been suggested that the adoption of the Eulerian elastic (natural) log-strain tensor

$$\mathbf{a} \equiv ln(\mathbf{v}) = ln(\mathbf{b}^{1/2}) = \tfrac{1}{2} ln(\mathbf{b}) \tag{4.1}$$

has certain desirable consequences. In this section, as in (3.19) above, $\{a_k, \hat{a}_k\}_{k=1}^3$ are taken to represent the principal values and corresponding directions of the natural strain **a**, with the principal strain differences

$$\mu_1 = a_2 - a_3 \; ; \quad \mu_2 = a_3 - a_1 \; ; \quad \mu_3 = a_1 - a_2,$$

as introduced in (3.20).

In view of the rate relation $(2.14)_1$ and the fourth order "symmetric product" and "**A**-bracket" tensors $(3.7)_1$ and (3.15), this particular strain measure is seen to evolve according to the corotational rate relation

---

[13] Refer to discussion immediately following (2.8).

$$\begin{aligned}
\overset{\circ}{\mathbf{a}} &= [\partial\mathbf{a}/\partial\mathbf{b}]\cdot\overset{\circ}{\mathbf{b}} \\
&= [\partial\mathbf{a}/\partial\mathbf{b}]\cdot\{\mathbf{bD}+\mathbf{Db}-2\mathbf{vD}_p\mathbf{v}\} \quad ; \quad [\mathbf{v}=\mathbf{b}^{1/2}] \\
&= [\partial\mathbf{a}/\partial\mathbf{b}]\cdot\{2\underset{\sim}{\mathbf{S}}_b\cdot\mathbf{D}-2\underset{\sim}{\mathbf{B}}_v\cdot\mathbf{D}_p\} \\
&= \{2[\partial\mathbf{a}/\partial\mathbf{b}]\circ\underset{\sim}{\mathbf{S}}_b\}\cdot\mathbf{D}-\{2[\partial\mathbf{a}/\partial\mathbf{b}]\circ\underset{\sim}{\mathbf{B}}_v\}\cdot\mathbf{D}_p\,, \\
\overset{\circ}{\mathbf{a}} &= \underset{\sim}{\mathbf{H}}_1\cdot\mathbf{D}-\underset{\sim}{\mathbf{H}}_2\cdot\mathbf{D}_p \quad ; \quad \left\{ \begin{array}{l} \underset{\sim}{\mathbf{H}}_1 \equiv 2[\partial\mathbf{a}/\partial\mathbf{b}]\circ\underset{\sim}{\mathbf{S}}_b\,, \\ \underset{\sim}{\mathbf{H}}_2 \equiv 2[\partial\mathbf{a}/\partial\mathbf{b}]\circ\underset{\sim}{\mathbf{B}}_v\,, \end{array}\right.
\end{aligned} \tag{4.2}$$

expressed in terms of new fourth order tensors $\underset{\sim}{\mathbf{H}}_1$ and $\underset{\sim}{\mathbf{H}}_2$. Close examination of the above development reveals that all second order tensors are symmetric, and that all fourth order tensors map symmetric second order tensors into symmetric tensors. The properties of these new fourth order tensors as linear operators restricted to the six (6) dimensional (vector) space of symmetric tensors have been considered previously by this author (Dashner, 1990), and are reviewed again in the Appendix. In particular, in (A.51) both are shown to be positive definite and symmetric so that

$$\left.\begin{array}{l} \mathbf{X}\bullet(\underset{\sim}{\mathbf{H}}_k\cdot\mathbf{X}) \left\{ \begin{array}{ll} =\mathbf{0}\ ; & \mathbf{X}=\mathbf{0}\,, \\ >\mathbf{0}\ ; & \text{otherwise}\,, \end{array}\right. \\ \underset{\sim}{\mathbf{H}}_k=\underset{\sim}{\mathbf{H}}_k^{\mathrm{T}} \;\Leftrightarrow\; \mathbf{X}\bullet(\underset{\sim}{\mathbf{H}}_k\cdot\mathbf{Y})=\mathbf{Y}\bullet(\underset{\sim}{\mathbf{H}}_k\cdot\mathbf{X})\,, \end{array}\right\} ;\quad k=1,2\,,$$

for any pair of symmetric tensors $(\mathbf{X},\mathbf{Y})$. With specific reference to (A.52) and (A.55), they are also shown to be non-singular and reduce to the identity mapping on the subspace of symmetric tensors which commute with $\mathbf{a}$. As in (A.21), this last property is formally expressed as

$$\mathbf{C}=\mathbf{C}^{\mathrm{T}}\in\mathcal{C}(\mathbf{a})\equiv\{\mathbf{X}\colon\mathbf{aX}=\mathbf{Xa}\} \;\Rightarrow\; \underset{\sim}{\mathbf{H}}_k\cdot\mathbf{C}=\mathbf{C} \quad ; \quad k=1,2\,. \tag{4.3}$$

More specifically, it is shown in $(\text{A.54})_1$ that, for any $\mathbf{X}=\mathbf{X}^{\mathrm{T}}$,

$$\begin{aligned}
\underset{\sim}{\mathbf{H}}_k\cdot\mathbf{X} = \mathbf{X} &+ h_k(\mu_1)(\hat{\mathbf{a}}_2\bullet\mathbf{X}\cdot\hat{\mathbf{a}}_3)(\hat{\mathbf{a}}_2\otimes\hat{\mathbf{a}}_3+\hat{\mathbf{a}}_3\otimes\hat{\mathbf{a}}_2) \\
&+ h_k(\mu_2)(\hat{\mathbf{a}}_3\bullet\mathbf{X}\cdot\hat{\mathbf{a}}_1)(\hat{\mathbf{a}}_3\otimes\hat{\mathbf{a}}_1+\hat{\mathbf{a}}_1\otimes\hat{\mathbf{a}}_3) \\
&+ h_k(\mu_3)(\hat{\mathbf{a}}_1\bullet\mathbf{X}\cdot\hat{\mathbf{a}}_2)(\hat{\mathbf{a}}_1\otimes\hat{\mathbf{a}}_2+\hat{\mathbf{a}}_2\otimes\hat{\mathbf{a}}_1) \quad ; \quad k=1,2\,,
\end{aligned} \tag{4.4}$$

expressed in terms of the scalar-valued coefficient functions

$$\begin{aligned}
h_1(\mu) &\equiv \frac{\mu}{\tanh(\mu)}-1=\tfrac{1}{3}\mu^2\left\{1-\tfrac{1}{15}\mu^2+\tfrac{2}{315}\mu^4+\ldots\right\}\ge\mathbf{0}\,, \\
h_2(\mu) &\equiv \frac{\mu}{\sinh(\mu)}-1=-\tfrac{1}{6}\mu^2\left\{1-\tfrac{7}{60}\mu^2+\tfrac{31}{2520}\mu^4+\ldots\right\}\le\mathbf{0}\,,
\end{aligned} \tag{4.5}$$

defined in (A.53).

Referencing (A.55) once again, it is also shown that (for any symmetric $\mathbf{C}$) the symmetric tensor difference $[(\underset{\sim}{\mathbf{H}}_k\cdot\mathbf{C})-\mathbf{C}]$ belongs to the *outer* (normal) *space* of the above defined tensor subspace $\mathcal{C}(\mathbf{a})$. This is formally expressed as

$$[\underset{\sim}{\mathbf{H}}_k\cdot\mathbf{C}-\mathbf{C}]\in\mathcal{C}^*(\mathbf{a})=\mathit{outer}[\mathcal{C}(\mathbf{a})]\equiv\{\mathbf{Y}\colon\mathbf{Y}\bullet\mathbf{X}=\mathbf{0}\ \forall\ \mathbf{X}\in\mathcal{C}(\mathbf{a})\} \quad ; \quad k=1,2\,.$$

This observation can now be combined with the result (A.28) which states that the equation

$$\mathbf{AX} - \mathbf{XA} = 2\underset{\sim}{\mathbf{B}}_{\mathrm{A}} \cdot \mathbf{X} = \mathbf{Y}$$

has a *unique* particular solution $\mathbf{X} \in \mathcal{C}^*(\mathbf{A})$ for every $\mathbf{Y} \in \mathcal{C}^*(\mathbf{A})$. Direct application of this for the specific choice $\mathbf{Y} = -\left[\underset{\sim}{\mathbf{H}}_k \cdot \mathbf{C} - \mathbf{C}\right] \in \mathcal{C}^*(\mathbf{a})$ leads to the immediate conclusion that there exists a unique solution $\mathbf{X} = \boldsymbol{\Omega}_{\mathrm{C}} \in \mathcal{C}^*(\mathbf{a})$ to the equation

$$\mathbf{a}\boldsymbol{\Omega}_{\mathrm{C}} - \boldsymbol{\Omega}_{\mathrm{C}}\mathbf{a} = 2\underset{\sim}{\mathbf{B}}_{\mathrm{a}} \cdot \boldsymbol{\Omega}_{\mathrm{C}} = -\left[\underset{\sim}{\mathbf{H}}_k \cdot \mathbf{C} - \mathbf{C}\right] \in \mathcal{C}^*(\mathbf{a}) \;;\quad k = 1,2\,, \tag{4.6}$$

for specified $\mathbf{C} = \mathbf{C}^{\mathrm{T}}$. Furthermore, the symmetry of $\left[\left(\underset{\sim}{\mathbf{H}}_k \cdot \mathbf{C}\right) - \mathbf{C}\right]$ and the established properties (3.8) guarantee that this solution is antisymmetric. Thus, **there is known to exist a unique antisymmetric tensor $\boldsymbol{\Omega}_{\mathrm{C}} \in \mathcal{C}^*(\mathbf{a})$ which satisfies**

$$\underset{\sim}{\mathbf{H}}_k \cdot \mathbf{C} = \mathbf{C} - \mathbf{a}\boldsymbol{\Omega}_{\mathrm{C}} + \boldsymbol{\Omega}_{\mathrm{C}}\mathbf{a} \;;\quad k = 1,2\,, \tag{4.7}$$

for any given symmetric C.

To ascertain the form of this solution, one need only examine the various terms of (4.6) after applying the eigenbasis expansions (A.44) and (4.4). Having done so, it is immediately apparent that this equation is satisfied if and only if

$$\begin{bmatrix} \mu_1(\hat{a}_2 \bullet \boldsymbol{\Omega}_{\mathrm{C}} \cdot \hat{a}_3) \\ \mu_2(\hat{a}_3 \bullet \boldsymbol{\Omega}_{\mathrm{C}} \cdot \hat{a}_1) \\ \mu_3(\hat{a}_1 \bullet \boldsymbol{\Omega}_{\mathrm{C}} \cdot \hat{a}_2) \end{bmatrix} = -\begin{bmatrix} h_k(\mu_1)(\hat{a}_2 \bullet \mathbf{C} \cdot \hat{a}_3) \\ h_k(\mu_2)(\hat{a}_3 \bullet \mathbf{C} \cdot \hat{a}_1) \\ h_k(\mu_3)(\hat{a}_1 \bullet \mathbf{C} \cdot \hat{a}_2) \end{bmatrix} \;;\quad k = 1,2\,,$$

which, in turn, serves to verify the antisymmetric tensor solution

$$\begin{aligned} \boldsymbol{\Omega}_{\mathrm{C}} = {} & \frac{h_k(\mu_1)}{\mu_1}[\hat{a}_2 \bullet \mathbf{C} \cdot \hat{a}_3][(\hat{a}_3 \otimes \hat{a}_2) - (\hat{a}_2 \otimes \hat{a}_3)] \\ & + \frac{h_k(\mu_2)}{\mu_2}[\hat{a}_3 \bullet \mathbf{C} \cdot \hat{a}_1][(\hat{a}_1 \otimes \hat{a}_3) - (\hat{a}_3 \otimes \hat{a}_1)] \\ & + \frac{h_k(\mu_3)}{\mu_3}[\hat{a}_1 \bullet \mathbf{C} \cdot \hat{a}_2][(\hat{a}_2 \otimes \hat{a}_1) - (\hat{a}_1 \otimes \hat{a}_2)] \in \mathcal{C}^*(\mathbf{A}) \;;\quad k = 1,2\,. \end{aligned} \tag{4.8}$$

If the stated conclusion that this belongs to $\mathcal{C}^*(\mathbf{a})$ is not initially obvious, it will become so after reviewing the development leading from (A.21) to (A.26) pertaining to the basic structure of the linear subspace $\mathcal{C}^*(\mathbf{a})$, and confirming, via (4.5), that the scalar coefficients $\left[h_k(\mu)/\mu\right]$ are each of order μ in the neighborhood of their apparent singularity at $\mu = 0$. To complete the characterization of this antisymmetric tensor solution, it is easily confirmed to have the associated axial vector $[\boldsymbol{\Omega}_{\mathrm{C}}\vec{u} = \vec{\boldsymbol{\Omega}}_{\mathrm{C}} \times \vec{u}]$

$$\vec{\boldsymbol{\Omega}}_{\mathrm{C}} = \frac{h_k(\mu_1)}{\mu_1}[\hat{a}_2 \bullet \mathbf{C} \cdot \hat{a}_3]\hat{a}_1 + \frac{h_k(\mu_2)}{\mu_2}[\hat{a}_3 \bullet \mathbf{C} \cdot \hat{a}_1]\hat{a}_2 + \frac{h_k(\mu_3)}{\mu_3}[\hat{a}_1 \bullet \mathbf{C} \cdot \hat{a}_2]\hat{a}_3 \;;\quad k = 1,2\,. \tag{4.9}$$

Making direct use of (4.7)- (4.9), the rate equation (4.2) for the natural strain can now be rewritten as

$$\overset{\circ}{\mathbf{a}} = \mathbf{D} - \mathbf{D}_{\mathrm{p}} - \mathbf{a}\left(\boldsymbol{\Omega}_{\mathrm{D}} - \boldsymbol{\Omega}_{\mathrm{D_p}}\right) + \left(\boldsymbol{\Omega}_{\mathrm{D}} - \boldsymbol{\Omega}_{\mathrm{D_p}}\right)\mathbf{a}$$

in terms of the spin rate tensors

$$\boldsymbol{\Omega}_{\mathrm{D}} = \boldsymbol{\Omega}_{\mathrm{D}} \quad \& \quad \boldsymbol{\Omega}_{\mathrm{D_p}} = \boldsymbol{\Omega}_{\mathrm{D_p}}$$

corresponding to the axial vectors

$$\vec{\Omega}_{D} = \frac{h_1(\mu_1)}{\mu_1}[\hat{a}_2 \bullet D \hat{a}_3]\hat{a}_1 + \frac{h_1(\mu_2)}{\mu_2}[\hat{a}_3 \bullet D \hat{a}_1]\hat{a}_2 + \frac{h_1(\mu_3)}{\mu_3}[\hat{a}_1 \bullet D \hat{a}_2]\hat{a}_3 ,$$

$$\vec{\Omega}_{D_p} = \frac{h_2(\mu_1)}{\mu_1}[\hat{a}_2 \bullet D_p \hat{a}_3]\hat{a}_1 + \frac{h_2(\mu_2)}{\mu_2}[\hat{a}_3 \bullet D_p \hat{a}_1]\hat{a}_2 + \frac{h_2(\mu_3)}{\mu_3}[\hat{a}_1 \bullet D_p \hat{a}_2]\hat{a}_3 .$$

With the definition of the new spin rate

$$\mathbf{W}^* \equiv \boldsymbol{\Omega}_D - \boldsymbol{\Omega}_{D_p} + \mathbf{W} ,$$

and the associated corotating time derivative

$$\overset{*}{\mathbf{A}} \equiv \dot{\mathbf{A}} + \mathbf{A}\mathbf{W}^* - \mathbf{W}^*\mathbf{A} ,$$

it is easily verified that

$$\begin{aligned}
\overset{\circ}{\mathbf{a}} &= \dot{\mathbf{a}} + \mathbf{a}\mathbf{W} - \mathbf{W}\mathbf{a} = \mathbf{D} - \mathbf{D}_p - \mathbf{a}(\boldsymbol{\Omega}_D - \boldsymbol{\Omega}_{D_p}) + (\boldsymbol{\Omega}_D - \boldsymbol{\Omega}_{D_p})\mathbf{a} ,\\
\Rightarrow \quad & \dot{\mathbf{a}} + \mathbf{a}(\boldsymbol{\Omega}_D - \boldsymbol{\Omega}_{D_p} + \mathbf{W}) - (\boldsymbol{\Omega}_D - \boldsymbol{\Omega}_{D_p} + \mathbf{W})\mathbf{a} = \mathbf{D} - \mathbf{D}_p \\
& \dot{\mathbf{a}} + \mathbf{a}\mathbf{W}^* - \mathbf{W}^*\mathbf{a} = \mathbf{D} - \mathbf{D}_p ,\\
& \Rightarrow \quad \overset{*}{\mathbf{a}} = \mathbf{D} - \mathbf{D}_p .
\end{aligned}$$

This establishes the existence of a *special* local reference frame relative to which the time derivative of the finite natural *elastic* strain tensor is equal to the difference of the observed material and plastic deformation rates. This result, with $\mathbf{D}_p = \mathbf{0}$, was demonstrated by Xiao *et al.* (1997) for the case where $\mathbf{a}$ represents the finite natural *total* log strain measured from some fixed material reference configuration.

While this result is mathematically interesting, it has no further relevance to the present development. As shall be demonstrated, it will prove more useful to have access to an expression for the log strain shadow rate. This is easily obtained with the aid of the previously established relations (3.23), (4.2), $(3.7)_2$, and $(3.13)_2$ as

$$\begin{aligned}
\overset{s}{\mathbf{a}} &\equiv \overset{\circ}{\mathbf{a}} + \mathbf{a}\boldsymbol{\Omega}_s - \boldsymbol{\Omega}_s\mathbf{a} \\
&= \underset{\sim}{\mathbf{H}}_1 \mathbf{D} - \underset{\sim}{\mathbf{H}}_2 \mathbf{D}_p + 2\underset{\sim}{\mathbf{A}}_a \boldsymbol{\Omega}_s \\
&= \underset{\sim}{\mathbf{H}}_1 \mathbf{D} - \underset{\sim}{\mathbf{H}}_2 \mathbf{D}_p - 2\underset{\sim}{\mathbf{A}}_a \left[\underset{\sim}{\mathbf{W}}_v (\mathbf{D} + \mathbf{D}_p)\right] \\
\overset{s}{\mathbf{a}} &= \left[\underset{\sim}{\mathbf{H}}_1 - 2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v\right]\mathbf{D} - \left[\underset{\sim}{\mathbf{H}}_2 + 2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v\right]\mathbf{D}_p .
\end{aligned}$$

Fortunately, this simplifies to

$$\boxed{\overset{s}{\mathbf{a}} = \underset{\sim}{\mathbf{H}}_2 \mathbf{D} - \underset{\sim}{\mathbf{H}}_1 \mathbf{D}_p} \tag{4.10}$$

as a consequence of the identity (A.56), *viz.*

$$\underset{\sim}{\mathbf{H}}_1 - \underset{\sim}{\mathbf{H}}_2 = 2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v \quad \Rightarrow \quad \left\{ \begin{array}{l} \underset{\sim}{\mathbf{H}}_1 - 2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v = \underset{\sim}{\mathbf{H}}_2 , \\ \underset{\sim}{\mathbf{H}}_2 + 2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v = \underset{\sim}{\mathbf{H}}_1 . \end{array} \right.$$

This final form is particularly interesting (and perhaps somewhat surprising) in light of the previously derived corotational rate relation (4.2).

5. Semi-lagrangian state variables

Before introducing the semi-Lagrangian state variable transformation, it is necessary to assemble a number of results pertaining to elastic deformation. First, the characteristic reference cell (*i.e.* bond structure) is materially embedded during a process of purely elastic deformation. Consistent with the rate equation (2.7) with $\mathcal{L}_p=\mathbf{0}$, and the relations (2.11), it therefore follows that such an elastic process, described by the local deformation gradient $\mathbf{F}$, induces the following changes in the elastic variables

$$\begin{aligned} \mathbf{F}_e &\rightarrow \mathbf{F}\mathbf{F}_e &&\Rightarrow \mathbf{F}_{e_2} \overset{e}{=} \mathbf{F}\mathbf{F}_{e_1}, \\ \mathbf{b} &\rightarrow \mathbf{F}\mathbf{b}\mathbf{F}^T &&\Rightarrow \mathbf{b}_2 \overset{e}{=} \mathbf{F}\mathbf{b}_1\mathbf{F}^T, \\ \mathbf{c} &\rightarrow \mathbf{F}^{-T}\mathbf{c}\mathbf{F}^{-1} &&\Rightarrow \mathbf{c}_2 \overset{e}{=} \mathbf{F}^{-T}\mathbf{c}_1\mathbf{F}^{-1}. \end{aligned} \tag{5.1}$$

[14]

In terms of the *stretch* and *unstretch* tensors (3.2), it is significant that the combination

$$\mathbf{M} \equiv \mathbf{u}_2\mathbf{F}\mathbf{v}_1 \tag{5.2}$$

is proper orthogonal since

$$\begin{aligned} \mathbf{M}\mathbf{M}^T &= (\mathbf{u}_2\mathbf{F}\mathbf{v}_1)(\mathbf{v}_1\mathbf{F}^T\mathbf{u}_2) \\ &= \mathbf{u}_2(\mathbf{F}\mathbf{b}_1\mathbf{F}^T)\mathbf{u}_2 \\ &\overset{e}{=} \mathbf{b}_2^{-1/2}\mathbf{b}_2\mathbf{b}_2^{-1/2} \\ \mathbf{M}\mathbf{M}^T &\overset{e}{=} \mathbf{I}. \end{aligned}$$

In fact, in the event of structural anisotropy, $\mathbf{M}$ fixes the new value of the cell orientation tensor through the orthogonal transformation expression

$$\begin{aligned} \mathbf{R}_{e_2} &= \mathbf{u}_2\mathbf{F}_{e_2} \ ; \quad [\mathbf{F}_e = \mathbf{v}\mathbf{R}_e \Leftrightarrow \mathbf{R}_e = \mathbf{u}\mathbf{F}_e] \\ &\overset{e}{=} \mathbf{u}_2(\mathbf{F}\mathbf{F}_{e_1}) = [\mathbf{u}_2\mathbf{F}](\mathbf{v}_1\mathbf{R}_{e_1}) = [\mathbf{u}_2\mathbf{F}\mathbf{v}_1]\mathbf{R}_{e_1} \\ \mathbf{R}_{e_2} &\overset{e}{=} \mathbf{M}\mathbf{R}_{e_1}. \end{aligned}$$

Moreover, since a rigid element rotation is a special case of purely elastic deformation, this last result, together with the established post-rotation transformations (2.13), guarantees that

$$\mathbf{F} \overset{e}{=} \mathbf{Q} \ \Rightarrow \ \mathbf{M} \overset{e}{=} \mathbf{Q} \ ; \quad \text{for each } \mathbf{Q} \in \Theta. \tag{5.3}$$

More generally, consider a smooth process of (perhaps inelastic) deformation imposed on a material element from a base ($t=t_i$) configuration in which its state is characterized by the initial values

$$\mathbf{v}(t_i) = \mathbf{v}_o \quad \& \quad \mathbf{u}(t_i) = \mathbf{u}_o = \mathbf{v}_o^{-1}.$$

Letting

$$\mathbf{F} = \mathbf{F}(t) \ ; \quad t \geq t_i \tag{5.4}$$

represent the total material deformation measured from this base configuration, it is clear that the initial condition $\mathbf{F}(t_i) = \mathbf{I}$ pertains. In view of (3.3) and (3.24), it is easily established that the above defined deformation measure

$$\mathbf{M}(t) = \mathbf{u}(t)\mathbf{F}(t)\mathbf{v}_o \ ; \quad t \geq t_i,$$

generally evolves according to the rate equation

[14] The notation "$\overset{e}{=}$" denotes equality under the special circumstance of purely *elastic* deformation.

$$\begin{aligned}
\dot{\mathbf{M}} &= \dot{\mathbf{u}}\mathbf{F}\mathbf{v}_o + \mathbf{u}\dot{\mathbf{F}}\mathbf{v}_o \,, \\
\dot{\mathbf{M}}\mathbf{M}^{-1} &= \left(\dot{\mathbf{u}}\mathbf{F}\mathbf{v}_o + \mathbf{u}\dot{\mathbf{F}}\mathbf{v}_o\right)\left(\mathbf{u}_o\mathbf{F}^{-1}\mathbf{v}\right) \\
&= \dot{\mathbf{u}}\mathbf{v} + \mathbf{u}\mathbf{L}\mathbf{v} \;\; ; \;\; \left[\mathbf{L} = \nabla\vec{\mathbf{v}} = \dot{\mathbf{F}}\mathbf{F}^{-1}\right] \\
&= \left(\mathbf{L}_s\mathbf{u} - \mathbf{u}\mathbf{L}\right)\mathbf{v} + \mathbf{u}\mathbf{L}\mathbf{v} \,,
\end{aligned}$$

$$\dot{\mathbf{M}}\mathbf{M}^{-1} = \mathbf{L}_s \quad \Rightarrow \quad \begin{cases} \dot{\mathbf{M}} = \left(\mathbf{D}_p + \boldsymbol{\Omega}_s + \mathbf{W}\right)\mathbf{M} \,, \\ \overset{o}{\mathbf{M}} = \left(\mathbf{D}_p + \boldsymbol{\Omega}_s\right)\mathbf{M} \,, \\ \overset{s}{\mathbf{M}} = \mathbf{D}_p\mathbf{M} \quad \Rightarrow \quad \overset{s}{\mathbf{M}} \overset{c}{=} \mathbf{0} \,, \end{cases}$$

subject to the initial condition $\mathbf{M}(t_i) = \mathbf{I}$. From this, it follows that the expressions

$$\begin{aligned}
&\mathbf{M}(t) \underset{\text{S.F.}}{\overset{c}{=}} \mathbf{I} \,, \\
&\mathbf{F}(t) \underset{\text{S.F.}}{\overset{c}{=}} \mathbf{v}(t)\mathbf{u}_o \,, \\
&\mathbf{v}(t) \underset{\text{S.F.}}{\overset{c}{=}} \mathbf{F}(t)\mathbf{v}_o = \mathbf{v}_o\mathbf{F}^{T}(t) \;\; ; \;\; \left[\mathbf{v} = \mathbf{v}^{T}\right] , \\
&\mathbf{u}(t) \underset{\text{S.F.}}{\overset{c}{=}} \mathbf{u}_o\mathbf{F}^{-1}(t) = \mathbf{F}^{-T}(t)\mathbf{u}_o \;\; ; \;\; \left[\mathbf{u} = \mathbf{u}^{T}\right] ,
\end{aligned} \tag{5.5}$$

accurately characterize the measurements of shadow frame observers during any ongoing process of purely elastic deformation commencing at $t = t_i$.

Now, let T_F represent the transformation operator, corresponding to a prescribed elastic deformation $\mathbf{F}$, which is appropriate for a tensor having the order and valence of a particular Eulerian state variable $\mathbf{q}$. The existence of this operator, as well as the group properties

$$\mathsf{T}_{F_2} \circ \mathsf{T}_{F_1} = \mathsf{T}_{F_2F_1} \quad \& \quad \mathsf{T}_I(\mathbf{q}) = \mathbf{q} \,,$$

can be inferred from the path independence of state variable evolution during purely elastic deformation. Moreover, since rigid element rotation is a special case of elastic deformation, this notation does not conflict with, but rather represents an extension of, the tensor transformation notation introduced in (2.4) and (2.5). This operator, together with the elastic *unstretching* tensor $\mathbf{u}$, can now be used to effect the transformation

$$\mathbf{p} \equiv \mathsf{T}_u(\mathbf{q}) \quad \Leftrightarrow \quad \mathbf{q} = \mathsf{T}_v(\mathbf{p}) \,. \tag{5.6}$$

The value assigned to this new variable p is clearly identical to the value that q would take following a purely elastic, rotation free, unstretching deformation returning the element to its instantaneous shadow configuration. Moreover, during a purely elastic deformation $\mathbf{F}$, each of these new variables is seen to evolve according to the rotation rule

$$\begin{aligned}
\mathbf{p}_2 &= \mathsf{T}_{u_2}(\mathbf{q}_2) \;\; ; \;\; \left[\mathbf{q}_2 \overset{c}{=} \mathsf{T}_F(\mathbf{q}_1)\right] \\
\mathbf{p}_2 &\overset{c}{=} \mathsf{T}_{u_2} \circ \mathsf{T}_F(\mathbf{q}_1) \\
&\overset{c}{=} \mathsf{T}_{u_2F}(\mathbf{q}_1) \;\; ; \;\; \left[\mathbf{q}_1 \overset{c}{=} \mathsf{T}_{v_1}(\mathbf{p}_1)\right] \\
&\overset{c}{=} \mathsf{T}_{u_2F} \circ \mathsf{T}_{v_1}(\mathbf{p}_1) \\
&\overset{c}{=} \mathsf{T}_{u_2Fv_1}(\mathbf{p}_1) \\
\mathbf{p}_2 &\overset{c}{=} \mathsf{T}_M(\mathbf{p}_1) \,,
\end{aligned} \tag{5.7}$$

in terms of proper orthogonal **M** from (5.2). In view of (5.3), it also follows that each Eulerian tensor state variable and its corresponding semi-Lagrangian variable transform in identical fashion during rigid element rotation[15]. Most important, however, is the fact that **each of these new variables is observed to remain constant in a shadow frame during a purely elastic deformation process.** This important result, which follows as a direct consequence of $(5.5)_1$ and (5.7), leads to the further conclusion that **each semi-Lagrangian variable evolves according to a frame invariant rate equation which can be cast in the form**

$$\overset{s}{\mathbf{p}}_\alpha = \pi_\alpha \;;\quad \alpha = 1,\ldots,N, \tag{5.8}$$

expressed in terms of an (inelastic) rate function π_α which vanishes identically for purely elastic deformation.

In order to simplify notation the symbols $\boldsymbol{p}$ and $\boldsymbol{\pi}$ shall henceforth be used to represent the tensor N-tuples

$$\boldsymbol{p} \equiv \{\mathbf{p}_\alpha\}_{\alpha=1}^{N} \quad \& \quad \boldsymbol{\pi} \equiv \{\pi_\alpha\}_{\alpha=1}^{N}, \tag{5.9}$$

so that the full set of semi-Lagrangian state variable evolution equations (5.8) may be collectively represented by $\overset{s}{\boldsymbol{p}} = \boldsymbol{\pi}$. Moreover, any interior (scalar) product involving 'collective' symbols of this type shall be interpreted as the full N-tuple interior product, *e.g.*

$$\left(\partial\Psi/\partial\boldsymbol{p}\right)\bullet\boldsymbol{\pi} = \sum_{\alpha=1}^{N}\left(\partial\Psi/\partial\mathbf{p}_\alpha\right)\bullet\pi_\alpha . \tag{5.10}$$

With the rate expressions $(3.28)_3$, (5.8), and the relations $(3.13)_2$ and (3.27), the general constitutive forms (2.1) can now be recast in the form

$$\begin{aligned}
&[\varphi,\tau] = \hat{\mathcal{R}}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \sim [\textit{response}\text{ functions}],\\
&[\mathbf{D}_p, \boldsymbol{\Omega}_e, \boldsymbol{\pi}] = \hat{\mathcal{E}}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \sim [\textit{evolution}\text{ functions}],\\
&\overset{s}{\mathbf{F}}_e = (\mathbf{D} - \boldsymbol{\Omega}_s)\mathbf{F}_e - \mathbf{F}_e\mathcal{L}_p \;;\quad \begin{cases} \mathcal{L}_p = \mathbf{R}_e^T(\mathbf{D}_p - \boldsymbol{\Omega}_e)\mathbf{R}_e, \\ \boldsymbol{\Omega}_s = -\underset{\sim}{\mathbf{W}}_v(\mathbf{D} + \mathbf{D}_p), \end{cases}\\
&\overset{s}{\boldsymbol{p}} = \boldsymbol{\pi}.
\end{aligned} \tag{5.11}$$

These modified forms are expressed in terms of the new semi-Lagrangian variables (5.6), the internal energy per unit reference volume

$$\varphi \equiv \rho_o\psi, \tag{5.12}$$

the Kirchhoff stress

$$\tau \equiv \left(\frac{\rho_o}{\rho}\right)\sigma, \tag{5.13}$$ [16]

---

[15] In this sense, each new semi-Lagrangian variable is still "Eulerian."

[16] This change of response variables obviously requires knowledge of the current mass density ρ in addition to the virgin state mass density ρ_o. As noted, this is accomplished through (3.31) in the event that plastic flow is incompressible. If plastic flow is not incompressible, then it would be necessary to independently identify the current scalar mass density as one of the inelastic state variables. Of course, its evolution equation

$$\dot{\rho} + \rho\, tr(\mathbf{D}) = 0$$

is easily recognized as the differential form of conservation of mass.

the cell placement decomposition constituents $\mathbf{F}_e = \mathbf{vR}_e$ from (3.2), the shadow frame vorticity $\mathbf{\Omega}_s$, and the inelastic state evolution functions $\hat{\mathcal{E}} = [\hat{\mathbf{D}}_p, \hat{\mathbf{\Omega}}_e, \hat{\boldsymbol{\pi}}]$. With reference to (2.4), (3.30), and the discussion immediately following (5.7), it is clear that each of the individual *response* and *evolution* functions $[\hat{\varphi}, \hat{\boldsymbol{\tau}}, \hat{\mathbf{D}}_p, \hat{\mathbf{\Omega}}_e, \hat{\boldsymbol{\pi}}]$ is subject to the 'Eulerian' invariance requirements

$$\begin{aligned} \mathsf{T}_Q[\hat{f}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A})] &= \hat{f}(\mathbf{QF}_e, \mathsf{T}_Q\boldsymbol{p}, \mathsf{T}_Q\boldsymbol{A}) \;; \quad \text{for each } \mathbf{Q} \in \Theta, \\ \hat{f}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) &= \hat{f}(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{p}, \boldsymbol{A}) \;; \quad \text{for each } \mathbf{Q} \in \wp, \end{aligned} \tag{5.14}$$

in terms of the appropriate rotational transformation operator.

In view of the invertible transformation

$$\mathbf{F}_e = \mathbf{vR}_e \quad \Leftrightarrow \quad \begin{cases} \mathbf{v} = \sqrt{\mathbf{F}_e\mathbf{F}_e^T}, \\ \mathbf{R}_e = (\mathbf{F}_e\mathbf{F}_e^T)^{-1/2}\mathbf{F}_e, \end{cases}$$

and the rate expressions (3.25) and (3.26), this theory can be recast in the equivalent general form

$$\begin{aligned} [\varphi, \boldsymbol{\tau}] &= \tilde{\mathcal{R}}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \sim [\textit{response}\,\text{functions}], \\ [\mathbf{D}_p, \mathbf{\Omega}_e, \boldsymbol{\pi}] &= \tilde{\mathcal{E}}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \sim [\textit{evolution}\,\text{functions}], \\ \overset{s}{\mathbf{v}} &= (\mathbf{D} - \mathbf{\Omega}_s)\mathbf{v} - \mathbf{vD}_p \;; \quad \mathbf{\Omega}_s = -\underset{\sim}{\mathbf{W}}_v \cdot (\mathbf{D} + \mathbf{D}_p), \\ \overset{s}{\mathbf{R}}_e &= \mathbf{\Omega}_e\mathbf{R}_e, \\ \overset{s}{\boldsymbol{p}} &= \boldsymbol{\pi}, \end{aligned} \tag{5.15}$$

in which each of the transformed response and evolution functions $[\tilde{\varphi}, \tilde{\boldsymbol{\tau}}, \tilde{\mathbf{D}}_p, \tilde{\mathbf{\Omega}}_e, \tilde{\boldsymbol{\pi}}]$ satisfies

$$\begin{aligned} \tilde{f}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) &= \hat{f}(\mathbf{vR}_e, \boldsymbol{p}, \boldsymbol{A}), \\ \hat{f}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) &= \tilde{f}[(\mathbf{F}_e\mathbf{F}_e^T)^{1/2}, (\mathbf{F}_e\mathbf{F}_e^T)^{-1/2}\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}], \end{aligned}$$

and is subject to the invariance requirements

$$\begin{aligned} \mathsf{T}_Q[\tilde{f}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A})] &= \tilde{f}(\mathbf{QvQ}^T, \mathbf{QR}_e, \mathsf{T}_Q\boldsymbol{p}, \mathsf{T}_Q\boldsymbol{A}) \;; \quad \text{for each } \mathbf{Q} \in \Theta, \\ \tilde{f}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) &= \tilde{f}(\mathbf{v}, \mathbf{R}_e\mathbf{Q}^T, \boldsymbol{p}, \boldsymbol{A}) \;; \quad \text{for each } \mathbf{Q} \in \wp, \end{aligned} \tag{5.16}$$

expressed in terms of the appropriate rotational transformation operator.

These equations can be recast in still other equivalent forms through the replacement of the *elastic stretch* tensor $\mathbf{v}$ with any convenient *elastic strain* tensor $\mathbf{e}$ defined through an invertible, isotropic, symmetric tensor function

$$\mathbf{e} = \mathcal{E}(\mathbf{v}) \quad \Leftrightarrow \quad \mathbf{v} = \mathcal{E}^{-1}(\mathbf{e}),$$

subject to the standard requirement that $\mathcal{E}(\mathbf{I}) = \mathbf{0}$. One such formulation would, of course, involve the *natural elastic strain* tensor

$$\mathbf{e} = \mathbf{a} = ln(\mathbf{v}) \quad \Leftrightarrow \quad \mathbf{v} = exp(\mathbf{a})$$

introduced in (4.1). After recalling the rate relation (4.10), this particular choice of strain measure gives rise to a general theory of the form

$$
\begin{aligned}
[\varphi, \tau] &= \widehat{\mathcal{R}}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \quad \sim \quad [\textit{response}\,\text{functions}], \\
[\mathbf{D}_p, \boldsymbol{\Omega}_e, \boldsymbol{\pi}] &= \widehat{\mathcal{E}}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \quad \sim \quad [\textit{evolution}\,\text{functions}], \\
\overset{s}{\mathbf{a}} &= \underset{\sim}{\mathbf{H}}_2 \mathbf{D} - \underset{\sim}{\mathbf{H}}_1 \mathbf{D}_p, \\
\overset{s}{\mathbf{R}}_e &= \boldsymbol{\Omega}_e \mathbf{R}_e, \\
\overset{s}{\boldsymbol{p}} &= \boldsymbol{\pi},
\end{aligned}
\tag{5.17}
$$

expressed in terms of the fourth order tensors $\underset{\sim}{\mathbf{H}}_1$ and $\underset{\sim}{\mathbf{H}}_2$ introduced in (4.2), and transformed response and evolution functions

$$
\begin{aligned}
\widehat{f}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) &= \widetilde{f}[\ln(\mathbf{v}), \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}], \\
\widetilde{f}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) &= \widehat{f}[\exp(\mathbf{a}), \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}],
\end{aligned}
$$

which are subject to the invariance constraints

$$
\begin{aligned}
\mathsf{T}_Q[\widehat{f}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A})] &= \widehat{f}(\mathbf{QaQ}^T, \mathbf{QR}_e, \mathsf{T}_Q\boldsymbol{p}, \mathsf{T}_Q\boldsymbol{A}) \; ; \quad \text{for each } \mathbf{Q} \in \Theta, \\
\widehat{f}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) &= \widehat{f}(\mathbf{a}, \mathbf{R}_e\mathbf{Q}^T, \boldsymbol{p}, \boldsymbol{A}) \; ; \quad \text{for each } \mathbf{Q} \in \wp .
\end{aligned}
\tag{5.18}
$$

In light of the discussion following (2.13), the gradient decomposition (3.2), and the rate expression $(3.26)_3$, it is also clear that *structurally isotropic* forms result from the replacement of the values

$$
\mathbf{R}_e = \mathbf{I} \; ; \quad \mathbf{F}_e = \mathbf{v} \; ; \quad \boldsymbol{\Omega}_e = \boldsymbol{0},
$$

into the above response and shadow rate expressions. It is a trivial matter to show that the constitutive forms (5.11) and (5.15) both reduce to

$$
\begin{aligned}
[\varphi, \tau] &= \widehat{\mathcal{R}}(\mathbf{v}, \boldsymbol{p}, \boldsymbol{A}) \quad \sim \quad [\textit{response}\,\text{functions}], \\
[\mathbf{D}_p, \boldsymbol{\pi}] &= \widehat{\mathcal{E}}(\mathbf{v}, \boldsymbol{p}, \boldsymbol{A}) \quad \sim \quad [\textit{evolution}\,\text{functions}], \\
\overset{s}{\mathbf{v}} &= (\mathbf{D} - \boldsymbol{\Omega}_s)\mathbf{v} - \mathbf{v}\mathbf{D}_p \; ; \quad \boldsymbol{\Omega}_s = -\underset{\sim}{\mathbf{W}}_v (\mathbf{D} + \mathbf{D}_p), \\
\overset{s}{\boldsymbol{p}} &= \boldsymbol{\pi},
\end{aligned}
\tag{5.19}
$$

while the equivalent natural strain formulation (5.17) simplifies to

$$
\begin{aligned}
[\varphi, \tau] &= \widehat{\mathcal{R}}(\mathbf{a}, \boldsymbol{p}, \boldsymbol{A}) \quad \sim \quad [\textit{response}\,\text{functions}], \\
[\mathbf{D}_p, \boldsymbol{\pi}] &= \widehat{\mathcal{E}}(\mathbf{a}, \boldsymbol{p}, \boldsymbol{A}) \quad \sim \quad [\textit{evolution}\,\text{functions}], \\
\overset{s}{\mathbf{a}} &= \underset{\sim}{\mathbf{H}}_2 \mathbf{D} - \underset{\sim}{\mathbf{H}}_1 \mathbf{D}_p, \\
\overset{s}{\boldsymbol{p}} &= \boldsymbol{\pi}.
\end{aligned}
\tag{5.20}
$$

Here, the invariance criteria $(5.16)_2$ and $(5.18)_2$ pertaining to prerotations of the reference cell have no further relevance. However, all response and evolution functions are still subject to the post-rotation requirements

$$
\left.
\begin{aligned}
\mathsf{T}_Q[\hat{f}(\mathbf{v}, \boldsymbol{p}, \boldsymbol{A})] &= \hat{f}(\mathbf{QvQ}^T, \mathsf{T}_Q\boldsymbol{p}, \mathsf{T}_Q\boldsymbol{A}) \\
\mathsf{T}_Q[\widehat{f}(\mathbf{a}, \boldsymbol{p}, \boldsymbol{A})] &= \widehat{f}(\mathbf{QaQ}^T, \mathsf{T}_Q\boldsymbol{p}, \mathsf{T}_Q\boldsymbol{A})
\end{aligned}
\right\} \; ; \quad \text{for each } \mathbf{Q} \in \Theta,
\tag{5.21}
$$

which insure the absolute *frame invariance* of the constitutive forms. Once again, it is important to emphasize that all of the above evolution functions vanish during any purely elastic deformation process. It is also significant to note {cf. Dashner (1986b), Sec. 3B} that the above response functions will exhibit no dependence on the *Rivlin-Ericksen* tensors $\boldsymbol{A}$ for materials that "respond continuously to continuous deformation stimuli." While this is not a characteristic of visco-elastic materials (which require smooth stimuli to elicit continuous response), it is for elastic and rate-independent elasto-plastic materials. For a much broader range of elastic visco-plastic materials, it is commonly assumed that the internal energy is determined by the instantaneous material state $(\mathbf{F}_e, \boldsymbol{p})$, but that stress response consists of a viscous component in addition to one that is state determined. The implications of specific material hypotheses of this type shall be considered in Section 7.

6. Elastic and non-elastic time rates

For metallic solids, the subset of purely elastic deformation processes is of particular interest. For this reason, it will prove useful to define an additive decomposition of the shadow time derivative into purely elastic and non-elastic components. For any scalar or tensor-valued function of state,

$$\Psi = \hat{\Psi}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\Psi}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \widehat{\Psi}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}), \tag{6.1}$$

these *elastic* and *non-elastic* shadow rate components are defined as

$$\overset{e}{\Psi} \equiv \left[\overset{s}{\Psi}\right]_{\mathbf{D}_p = \Omega_e = \boldsymbol{\pi} = \boldsymbol{0}} \quad \& \quad \overset{n}{\Psi} \equiv \overset{s}{\Psi} - \overset{e}{\Psi}, \tag{6.2}$$

the former being defined as the shadow rate with all inelastic deformation mechanisms deactivated.

To facilitate evaluation of these time derivative components, the rate expressions from the equation sets (5.11), (5.15), (5.17), are now reexamined and recast in terms of their elastic and non-elastic components:

$$\begin{aligned}
&\overset{s}{\boldsymbol{p}} = \overset{e}{\boldsymbol{p}} + \overset{n}{\boldsymbol{p}} \;; \quad \begin{cases} \overset{e}{\boldsymbol{p}} = \boldsymbol{0}, \\ \overset{n}{\boldsymbol{p}} = \boldsymbol{\pi}, \end{cases} \\
&\overset{s}{\mathbf{R}}_e = \overset{e}{\mathbf{R}}_e + \overset{n}{\mathbf{R}}_e \;; \quad \begin{cases} \overset{e}{\mathbf{R}}_e = \boldsymbol{0}, \\ \overset{n}{\mathbf{R}}_e = \Omega_e \mathbf{R}_e, \end{cases} \\
&\overset{s}{\mathbf{v}} = \overset{e}{\mathbf{v}} + \overset{n}{\mathbf{v}} \;; \quad \begin{cases} \overset{e}{\mathbf{v}} = (\mathbf{D} + \underset{\sim}{\mathbf{W}}_v \mathbf{D})\mathbf{v} = \mathbf{v}(\mathbf{D} - \underset{\sim}{\mathbf{W}}_v \mathbf{D}), \\ \overset{n}{\mathbf{v}} = (\underset{\sim}{\mathbf{W}}_v \mathbf{D}_p)\mathbf{v} - \mathbf{v}\mathbf{D}_p = -\left[\mathbf{v}(\underset{\sim}{\mathbf{W}}_v \mathbf{D}_p) + \mathbf{D}_p \mathbf{v}\right], \end{cases} \\
&\overset{s}{\mathbf{F}}_e = \overset{e}{\mathbf{F}}_e + \overset{n}{\mathbf{F}}_e \;; \quad \begin{cases} \overset{e}{\mathbf{F}}_e = (\mathbf{D} + \underset{\sim}{\mathbf{W}}_v \mathbf{D})\mathbf{F}_e, \\ \overset{n}{\mathbf{F}}_e = (\underset{\sim}{\mathbf{W}}_v \mathbf{D}_p)\mathbf{F}_e - \mathbf{F}_e\left[\mathbf{R}_e^T(\mathbf{D}_p - \Omega_e)\mathbf{R}_e\right], \end{cases} \\
&\overset{s}{\mathbf{a}} = \overset{e}{\mathbf{a}} + \overset{n}{\mathbf{a}} \;; \quad \begin{cases} \overset{e}{\mathbf{a}} = \underset{\sim}{\mathbf{H}}_2 \mathbf{D}, \\ \overset{n}{\mathbf{a}} = -\underset{\sim}{\mathbf{H}}_1 \mathbf{D}_p. \end{cases}
\end{aligned} \tag{6.3}$$

From this, it is immediately clear that generally

$$\overset{s}{\Psi} = \overset{e}{\Psi} + \overset{n}{\Psi} \;; \quad \begin{cases} \overset{e}{\Psi} = \dfrac{\partial\hat{\Psi}}{\partial \mathbf{F}_e}\bullet \overset{e}{\mathbf{F}}_e = \dfrac{\partial\tilde{\Psi}}{\partial \mathbf{v}}\bullet\overset{e}{\mathbf{v}} = \dfrac{\partial\overset{\frown}{\Psi}}{\partial \mathbf{a}}\bullet \overset{e}{\mathbf{a}}, \\ \overset{n}{\Psi} = \begin{cases} \dfrac{\partial\hat{\Psi}}{\partial \mathbf{F}_e}\bullet \overset{n}{\mathbf{F}}_e + \dfrac{\partial\hat{\Psi}}{\partial \boldsymbol{p}}\bullet \boldsymbol{\pi}, \\ \dfrac{\partial\tilde{\Psi}}{\partial \mathbf{v}}\bullet \overset{n}{\mathbf{v}} + \dfrac{\partial\tilde{\Psi}}{\partial \mathbf{R}_e}\bullet (\boldsymbol{\Omega}_e\mathbf{R}_e) + \dfrac{\partial\tilde{\Psi}}{\partial \boldsymbol{p}}\bullet \boldsymbol{\pi}, \\ \dfrac{\partial\overset{\frown}{\Psi}}{\partial \mathbf{a}}\bullet \overset{n}{\mathbf{a}} + \dfrac{\partial\overset{\frown}{\Psi}}{\partial \mathbf{R}_e}\bullet (\boldsymbol{\Omega}_e\mathbf{R}_e) + \dfrac{\partial\overset{\frown}{\Psi}}{\partial \boldsymbol{p}}\bullet \boldsymbol{\pi}, \end{cases} \end{cases} \tag{6.4}$$

for functions of this type.

For the thermodynamic considerations to follow, it will also prove useful to have ready access to expressions for the (shadow frame) time derivative of a frame invariant, scalar-valued state function expressed in either of the alternative forms

$$\mathsf{f} = \hat{\mathsf{f}}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\mathsf{f}}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \overset{\frown}{\mathsf{f}}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}).$$

The derivative expansions below are straightforward, making use of the above listed state variable rate equations, the symmetry of the fourth order tensors $\underset{\sim}{\mathbf{W}}_v$, $\underset{\sim}{\mathbf{H}}_1$ and $\underset{\sim}{\mathbf{H}}_2$, and the well known properties

$$\mathbf{X}\bullet\mathbf{YZ} = \mathbf{XZ}^T\bullet\mathbf{Y} = \mathbf{Y}^T\mathbf{X}\bullet\mathbf{Y},$$
$$\mathbf{X} = \mathbf{X}^T \;\Rightarrow\; \mathbf{Z}\bullet\mathbf{X} = [\mathbf{Z}]_S\bullet\mathbf{X},$$
$$\mathbf{X} = -\mathbf{X}^T \;\Rightarrow\; \mathbf{Z}\bullet\mathbf{X} = [\mathbf{Z}]_A\bullet\mathbf{X},$$

involving the inner product $\mathbf{X}\bullet\mathbf{Y} \equiv tr(\mathbf{XY}^T)$ of second order tensors.

$$\begin{aligned} \dot{\mathsf{f}} = \overset{s}{\mathsf{f}} &= \frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\bullet\overset{s}{\mathbf{F}}_e + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\overset{s}{\boldsymbol{p}} \\ &= \frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\bullet\left[(\mathbf{D}-\boldsymbol{\Omega}_s)\mathbf{F}_e - \mathbf{F}_e\mathcal{L}_p\right] + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\boldsymbol{\pi} \\ &= \left(\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right)\bullet(\mathbf{D}-\boldsymbol{\Omega}_s) - \left(\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right)\bullet\left[\mathbf{R}_e^T(\mathbf{D}_p-\boldsymbol{\Omega}_e)\mathbf{R}_e\right] + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\boldsymbol{\pi} \\ &= \left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_S\bullet\mathbf{D} - \left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_A\bullet\boldsymbol{\Omega}_s - \left[\mathbf{R}_e\left(\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right)\mathbf{R}_e^T\right]\bullet(\mathbf{D}_p-\boldsymbol{\Omega}_e) + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\boldsymbol{\pi} \\ &= \left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_S\bullet\mathbf{D} + \left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_A\bullet\underset{\sim}{\mathbf{W}}_v(\mathbf{D}+\mathbf{D}_p) - \left[\mathbf{R}_e\left(\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right)\mathbf{R}_e^T\right]_S\bullet\mathbf{D}_p + \left[\mathbf{R}_e\left(\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right)\mathbf{R}_e^T\right]_A\bullet\boldsymbol{\Omega}_e + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\boldsymbol{\pi} \\ &= \left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_S\bullet\mathbf{D} + \left(\underset{\sim}{\mathbf{W}}_v\left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_A\right)\bullet(\mathbf{D}+\mathbf{D}_p) - \left(\mathbf{R}_e\left[\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right]_S\mathbf{R}_e^T\right)\bullet\mathbf{D}_p + \left(\mathbf{R}_e\left[\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right]_A\mathbf{R}_e^T\right)\bullet\boldsymbol{\Omega}_e + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\boldsymbol{\pi}, \end{aligned}$$

$$\boxed{\begin{aligned} \dot{\mathsf{f}} = &\left\{\left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_S + \underset{\sim}{\mathbf{W}}_v\left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_A\right\}\bullet\mathbf{D} \\ &- \left\{\mathbf{R}_e\left[\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right]_S\mathbf{R}_e^T - \underset{\sim}{\mathbf{W}}_v\left[\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_A\right\}\bullet\mathbf{D}_p \\ &+ \left\{\mathbf{R}_e\left[\mathbf{F}_e^T\frac{\partial\hat{\mathsf{f}}}{\partial\mathbf{F}_e}\right]_A\mathbf{R}_e^T\right\}\bullet\boldsymbol{\Omega}_e + \frac{\partial\hat{\mathsf{f}}}{\partial\boldsymbol{p}}\bullet\boldsymbol{\pi} \end{aligned}}\;.$$

$$\begin{aligned}
\dot{\mathrm{f}} = \overset{\mathrm{s}}{\mathrm{f}} &= \frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}} \bullet \overset{\mathrm{s}}{\mathbf{v}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}} \bullet \overset{\mathrm{s}}{\mathbf{R}}_{\mathrm{e}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \overset{\mathrm{s}}{\boldsymbol{p}} \\
&= \frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}} \bullet [(\mathbf{D} - \boldsymbol{\Omega}_{\mathrm{s}})\mathbf{v} - \mathbf{v}\mathbf{D}_{\mathrm{p}}] + \frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}} \bullet (\boldsymbol{\Omega}_{\mathrm{e}}\mathbf{R}_{\mathrm{e}}) + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi} \\
&= \left(\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right) \bullet (\mathbf{D} - \boldsymbol{\Omega}_{\mathrm{s}}) - \left(\mathbf{v}\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\right) \bullet \mathbf{D}_{\mathrm{p}} + \left(\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right) \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi} \\
&= \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} \bullet \mathbf{D} - \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{A}} \bullet \boldsymbol{\Omega}_{\mathrm{s}} - \left[\mathbf{v}\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\right]_{\mathrm{S}} \bullet \mathbf{D}_{\mathrm{p}} + \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{A}} \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi} \\
&= \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} \bullet \mathbf{D} + \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{A}} \bullet \underset{\sim}{\mathbf{W}}_{\mathrm{v}} (\mathbf{D} + \mathbf{D}_{\mathrm{p}}) - \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} \bullet \mathbf{D}_{\mathrm{p}} + \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{A}} \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi} \\
&= \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} \bullet \mathbf{D} + \left(\underset{\sim}{\mathbf{W}}_{\mathrm{v}} \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{A}}\right) \bullet (\mathbf{D} + \mathbf{D}_{\mathrm{p}}) - \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} \bullet \mathbf{D}_{\mathrm{p}} + \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{A}} \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi},
\end{aligned}$$

$$\boxed{\begin{aligned}
\dot{\mathrm{f}} = &\left\{\left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} + \underset{\sim}{\mathbf{W}}_{\mathrm{v}} \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{A}}\right\} \bullet \mathbf{D} \\
&- \left\{\left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} - \underset{\sim}{\mathbf{W}}_{\mathrm{v}} \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{A}}\right\} \bullet \mathbf{D}_{\mathrm{p}} \\
&+ \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{A}} \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \tilde{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi}
\end{aligned}} .$$

$$\begin{aligned}
\dot{\mathrm{f}} = \overset{\mathrm{s}}{\mathrm{f}} &= \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}} \bullet \overset{\mathrm{s}}{\mathbf{a}} + \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}} \bullet \overset{\mathrm{s}}{\mathbf{R}}_{\mathrm{e}} + \frac{\partial \hat{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \overset{\mathrm{s}}{\boldsymbol{p}} \\
&= \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}} \bullet (\underset{\sim}{\mathbf{H}}_2 \mathbf{D} - \underset{\sim}{\mathbf{H}}_1 \mathbf{D}_{\mathrm{p}}) + \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}} \bullet (\boldsymbol{\Omega}_{\mathrm{e}}\mathbf{R}_{\mathrm{e}}) + \frac{\partial \hat{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi} \\
&= \left(\frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}}\right) \bullet \underset{\sim}{\mathbf{H}}_2 \mathbf{D} - \left(\frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}}\right) \bullet \underset{\sim}{\mathbf{H}}_1 \mathbf{D}_{\mathrm{p}} + \left(\frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right) \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \hat{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi},
\end{aligned}$$

$$\boxed{\dot{\mathrm{f}} = \left(\underset{\sim}{\mathbf{H}}_2 \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}}\right) \bullet \mathbf{D} - \left(\underset{\sim}{\mathbf{H}}_1 \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}}\right) \bullet \mathbf{D}_{\mathrm{p}} + \left[\frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{R}_{\mathrm{e}}}\mathbf{R}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{A}} \bullet \boldsymbol{\Omega}_{\mathrm{e}} + \frac{\partial \hat{\mathrm{f}}}{\partial \boldsymbol{p}} \bullet \boldsymbol{\pi}} .$$

These various alternative forms can all be expressed as

$$\boxed{\dot{\mathrm{f}} = \overset{\mathrm{s}}{\mathrm{f}} = [\nabla_{\mathrm{D}} \mathrm{f}] \bullet \mathbf{D} + [\nabla_{\mathrm{D_p}} \mathrm{f}] \bullet \mathbf{D}_{\mathrm{p}} + [\nabla_{\Omega_{\mathrm{e}}} \mathrm{f}] \bullet \boldsymbol{\Omega}_{\mathrm{e}} + [\nabla_{\boldsymbol{\pi}} \mathrm{f}] \bullet \boldsymbol{\pi}} \tag{6.5}$$

in terms of the 'gradient-type' tensor coefficients

$$\nabla_{\mathrm{D}} \mathrm{f} = \left\{\left[\frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{F}_{\mathrm{e}}}\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{S}} + \underset{\sim}{\mathbf{W}}_{\mathrm{v}} \left[\frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{F}_{\mathrm{e}}}\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\right]_{\mathrm{A}}\right\} = \left\{\left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{S}} + \underset{\sim}{\mathbf{W}}_{\mathrm{v}} \left[\frac{\partial \tilde{\mathrm{f}}}{\partial \mathbf{v}}\mathbf{v}\right]_{\mathrm{A}}\right\} = \underset{\sim}{\mathbf{H}}_2 \frac{\partial \hat{\mathrm{f}}}{\partial \mathbf{a}} ,$$

$$\nabla_{D_p} f = -\left\{ \mathbf{R}_e \left[\mathbf{F}_e^T \frac{\partial \hat{f}}{\partial \mathbf{F}_e} \right]_S \mathbf{R}_e^T - \underset{\sim}{\mathbf{W}}_v \cdot \left[\frac{\partial \hat{f}}{\partial \mathbf{F}_e} \mathbf{F}_e^T \right]_A \right\} = -\left\{ \left[\frac{\partial \tilde{f}}{\partial \mathbf{v}} \mathbf{v} \right]_S - \underset{\sim}{\mathbf{W}}_v \cdot \left[\frac{\partial \tilde{f}}{\partial \mathbf{v}} \mathbf{v} \right]_A \right\} = -\left(\underset{\sim}{\mathbf{H}}_1 \cdot \frac{\partial \widehat{f}}{\partial \mathbf{a}} \right),$$

$$\nabla_{\Omega_e} f = \mathbf{R}_e \left[\mathbf{F}_e^T \frac{\partial \hat{f}}{\partial \mathbf{F}_e} \right]_A \mathbf{R}_e^T = \left[\frac{\partial \tilde{f}}{\partial \mathbf{R}_e} \mathbf{R}_e^T \right]_A = \left[\frac{\partial \widehat{f}}{\partial \mathbf{R}_e} \mathbf{R}_e^T \right]_A, \tag{6.6}$$

$$\nabla_{\pi} f = \frac{\partial \hat{f}}{\partial \boldsymbol{p}} = \frac{\partial \tilde{f}}{\partial \boldsymbol{p}} = \frac{\partial \widehat{f}}{\partial \boldsymbol{p}}.$$

In terms of the above defined gradient coefficients, the elastic and non-elastic components for this scalar time rate take the form

$$\dot{f} = \overset{s}{f} = \overset{e}{f} + \overset{n}{f} \; ; \quad \left\{ \begin{aligned} \overset{e}{f} &= [\nabla_D f] \bullet \mathbf{D}, \\ \overset{n}{f} &= [\nabla_{D_p} f] \bullet \mathbf{D}_p + [\nabla_{\Omega_e} f] \bullet \boldsymbol{\Omega}_e + [\nabla_{\pi} f] \bullet \boldsymbol{\pi}. \end{aligned} \right. \tag{6.7}$$

7. Basic thermodynamic considerations

Basic thermodynamic considerations (within the present purely mechanical context) begin with the dissipation inequality

$$\dot{\gamma} \equiv \boldsymbol{\tau} \bullet \mathbf{D} - \dot{\varphi} \geq 0, \tag{7.1}$$

expressed in terms of the response pair

$$[\varphi, \boldsymbol{\tau}] = [\rho_o \psi, (\rho_o/\rho) \boldsymbol{\sigma}]$$

consisting of the internal energy per unit reference volume and the Kirchhoff stress. This purely mechanical version of the Clausius-Duhem inequality asserts that, at any given moment, and measured per unit of virgin state volume, *the rate of increase of internal energy cannot exceed the instantaneous power generated by the local stress*. This excess stress power not being absorbed as internal energy is known as the rate of *mechanical dissipation* (per unit of virgin state volume) $\dot{\gamma}$.

With reference to the compiled constitutive forms $(5.11)_1$, $(5.15)_1$ and $(5.17)_1$, and the scalar-valued state function time-rate expansion (6.5), it immediately follows that

$$\dot{\varphi} = \overset{s}{\varphi} = [\nabla_D \varphi] \bullet \mathbf{D} + [\nabla_{D_p} \varphi] \bullet \mathbf{D}_p + [\nabla_{\Omega_e} \varphi] \bullet \boldsymbol{\Omega}_e + [\nabla_{\pi} \varphi] \bullet \boldsymbol{\pi} + [\nabla_{A} \varphi] \bullet \overset{s}{\boldsymbol{A}},$$

expressed in terms of the inelastic rates $\mathbf{D}_p$, $\boldsymbol{\Omega}_e$ and $\boldsymbol{\pi}$, and the energy gradients (6.6) to which the new definition

$$\nabla_{A} \varphi \equiv \frac{\partial \varphi}{\partial \boldsymbol{A}} = \left\{ \frac{\partial \varphi}{\partial \mathbf{A}_k} \right\}_{k=1}^{M}$$

has been appended. Substitution of this into the dissipation inequality (7.1) yields

$$\dot{\gamma} = \{ \boldsymbol{\tau} - [\nabla_D \varphi] \} \bullet \mathbf{D} - [\nabla_{D_p} \varphi] \bullet \mathbf{D}_p - [\nabla_{\Omega_e} \varphi] \bullet \boldsymbol{\Omega}_e - [\nabla_{\pi} \varphi] \bullet \boldsymbol{\pi} - [\nabla_{A} \varphi] \bullet \overset{s}{\boldsymbol{A}} \geq 0.$$

By introducing the Kirchhoff stress decomposition

$$\boldsymbol{\tau} = \boldsymbol{\tau}_e + \boldsymbol{\tau}_f \quad \Rightarrow \quad \boldsymbol{\tau}_f = \boldsymbol{\tau} - \boldsymbol{\tau}_e, \tag{7.2}$$

in which the *elastic stress* component $\boldsymbol{\tau}_e$ is defined as the "D-gradient" $(6.6)_1$ of internal energy

$$\boldsymbol{\tau}_e \equiv \boldsymbol{\nabla}_D \varphi , \tag{7.3}$$

and direct specification of a stress response function for $\boldsymbol{\tau}$ is replaced with one for the so-called 'viscous' or 'frictional' stress $\boldsymbol{\tau}_f$, this inequality takes the equivalent form

$$\dot{\gamma} = \boldsymbol{\tau}_f \bullet \mathbf{D} + \boldsymbol{\tau}_d \bullet \mathbf{D}_p + \boldsymbol{\Gamma}_c \bullet \boldsymbol{\Omega}_c + \boldsymbol{\Gamma}_\pi \bullet \boldsymbol{\pi} + \boldsymbol{\Gamma}_A \bullet \overset{s}{\dot{\boldsymbol{A}}} \geq 0 ,$$

expressed in terms of the above defined stress components and the 'stress-like' tensor coefficients

$$\boldsymbol{\tau}_d \equiv -\left[\boldsymbol{\nabla}_{D_p}\varphi\right]; \ \boldsymbol{\Gamma}_c \equiv -\left[\boldsymbol{\nabla}_{\Omega_c}\varphi\right]; \ \boldsymbol{\Gamma}_\pi \equiv -\left[\boldsymbol{\nabla}_{\boldsymbol{\pi}}\varphi\right]; \ \boldsymbol{\Gamma}_A \equiv -\left[\boldsymbol{\nabla}_{\boldsymbol{A}}\varphi\right]. \tag{7.4}$$

The fact that this must hold for all possible deformation processes, through any accessible state, places useful restrictions on the various constitutive functions. Before investigating a number of important special cases, the essential elements of this general theoretical structure are collected and presented in the following compilation:

RESPONSE FUNCTIONS & RELATIONS:

$$\varphi = \begin{cases} \hat{\varphi}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \tilde{\varphi}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \widehat{\varphi}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \end{cases} ; \quad \begin{cases} \boldsymbol{\tau}_e \equiv \boldsymbol{\nabla}_D \varphi = \begin{cases} \left[\dfrac{\partial \hat{\varphi}}{\partial \mathbf{F}_e}\mathbf{F}_e^T\right]_S + \underset{\sim}{\mathbf{W}}_v \cdot \left[\dfrac{\partial \hat{\varphi}}{\partial \mathbf{F}_e}\mathbf{F}_e^T\right]_A , \\ \left[\dfrac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_S + \underset{\sim}{\mathbf{W}}_v \cdot \left[\dfrac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_A , \\ \underset{\sim}{\mathbf{H}}_2 \cdot \dfrac{\partial \widehat{\varphi}}{\partial \mathbf{a}} , \end{cases} \\ \boldsymbol{\tau}_d \equiv -\left[\boldsymbol{\nabla}_{D_p}\varphi\right] = \begin{cases} \mathbf{R}_e\left[\mathbf{F}_e^T \dfrac{\partial \hat{\varphi}}{\partial \mathbf{F}_e}\right]_S \mathbf{R}_e^T - \underset{\sim}{\mathbf{W}}_v \cdot \left[\dfrac{\partial \hat{\varphi}}{\partial \mathbf{F}_e}\mathbf{F}_e^T\right]_A , \\ \left[\dfrac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_S - \underset{\sim}{\mathbf{W}}_v \cdot \left[\dfrac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_A , \\ \underset{\sim}{\mathbf{H}}_1 \cdot \dfrac{\partial \widehat{\varphi}}{\partial \mathbf{a}} , \end{cases} \\ \boldsymbol{\Gamma}_c \equiv -\left[\boldsymbol{\nabla}_{\Omega_c}\varphi\right] = -\mathbf{R}_e\left[\mathbf{F}_e^T \dfrac{\partial \hat{\varphi}}{\partial \mathbf{F}_e}\right]_A \mathbf{R}_e^T = -\left[\dfrac{\partial \tilde{\varphi}}{\partial \mathbf{R}_e}\mathbf{R}_e^T\right]_A = -\left[\dfrac{\partial \widehat{\varphi}}{\partial \mathbf{R}_e}\mathbf{R}_e^T\right]_A , \\ \boldsymbol{\Gamma}_\pi \equiv -\left[\boldsymbol{\nabla}_{\boldsymbol{\pi}}\varphi\right] = -\dfrac{\partial \hat{\varphi}}{\partial \boldsymbol{p}} = -\dfrac{\partial \tilde{\varphi}}{\partial \boldsymbol{p}} = -\dfrac{\partial \widehat{\varphi}}{\partial \boldsymbol{p}} = -\left\{\dfrac{\partial \varphi}{\partial \mathbf{p}_\alpha}\right\}_{\alpha=1}^{N} , \\ \boldsymbol{\Gamma}_A \equiv -\left[\boldsymbol{\nabla}_{\boldsymbol{A}}\varphi\right] = -\dfrac{\partial \hat{\varphi}}{\partial \boldsymbol{A}} = -\dfrac{\partial \tilde{\varphi}}{\partial \boldsymbol{A}} = -\dfrac{\partial \widehat{\varphi}}{\partial \boldsymbol{A}} = -\left\{\dfrac{\partial \varphi}{\partial \mathbf{A}_k}\right\}_{k=1}^{M} , \end{cases}$$

$$\boldsymbol{\tau}_f = \begin{cases} \hat{\boldsymbol{\tau}}_f(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \tilde{\boldsymbol{\tau}}_f(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \widehat{\boldsymbol{\tau}}_f(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \end{cases} , \tag{7.5}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}_e + \boldsymbol{\tau}_f .$$

EVOLUTION FUNCTIONS & RELATIONS:

$$\mathbf{D}_p = \begin{cases} \hat{\mathbf{D}}_p(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \tilde{\mathbf{D}}_p(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \widehat{\mathbf{D}}_p(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \end{cases} ; \quad \boldsymbol{\Omega}_e = \begin{cases} \hat{\boldsymbol{\Omega}}_e(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \tilde{\boldsymbol{\Omega}}_e(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \widehat{\boldsymbol{\Omega}}_e(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \end{cases} ; \quad \boldsymbol{\pi} = \begin{cases} \hat{\boldsymbol{\pi}}(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \tilde{\boldsymbol{\pi}}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \\ \widehat{\boldsymbol{\pi}}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}, \boldsymbol{A}) \end{cases},$$

$$\overset{s}{\mathbf{F}}_e = (\mathbf{D} - \boldsymbol{\Omega}_s)\mathbf{F}_e - \mathbf{F}_e \mathcal{L}_p \; ; \quad \begin{cases} \mathcal{L}_p = \mathbf{R}_e^T(\mathbf{D}_p - \boldsymbol{\Omega}_e)\mathbf{R}_e , \\ \boldsymbol{\Omega}_s = \underset{\sim}{\mathbf{W}}_u \cdot (\mathbf{D} + \mathbf{D}_p) = -\underset{\sim}{\mathbf{W}}_v \cdot (\mathbf{D} + \mathbf{D}_p) , \end{cases}$$

$$\begin{aligned} \overset{s}{\mathbf{v}} &= (\mathbf{D} - \boldsymbol{\Omega}_s)\mathbf{v} - \mathbf{v}\mathbf{D}_p , \\ \overset{s}{\mathbf{a}} &= \underset{\sim}{\mathbf{H}}_2 \cdot \mathbf{D} - \underset{\sim}{\mathbf{H}}_1 \cdot \mathbf{D}_p , \\ \overset{s}{\mathbf{R}}_e &= \boldsymbol{\Omega}_e \mathbf{R}_e , \\ \overset{s}{\boldsymbol{p}} &= \boldsymbol{\pi} . \end{aligned} \tag{7.6}$$

DISSIPATION INEQUALITY:

$$\dot{\gamma} = \boldsymbol{\tau}_f \bullet \mathbf{D} + \boldsymbol{\tau}_d \bullet \mathbf{D}_p + \boldsymbol{\Gamma}_e \bullet \boldsymbol{\Omega}_e + \boldsymbol{\Gamma}_\pi \bullet \boldsymbol{\pi} + \boldsymbol{\Gamma}_{\boldsymbol{A}} \bullet \overset{s}{\boldsymbol{A}} \geq \boldsymbol{0} . \tag{7.7}$$

Theoretical consequences of specific material hypotheses

• ***Internal energy is a state property***. This material hypothesis is appropriate for materials whose internal energy is independent of the deformation rates which comprise the Rivlin-Ericksen set $\boldsymbol{A}$. In view of the universality of this assumption in the constitutive modeling of solids, it shall henceforth be adopted without further reference or comment. The immediate mathematical consequence of this is that the Rivlin-Ericksen stress coefficients $\boldsymbol{\Gamma}_{\boldsymbol{A}}$ vanish, *i.e.*

$$\frac{\partial \varphi}{\partial \boldsymbol{A}} = \boldsymbol{0} \quad \Rightarrow \quad \begin{cases} \varphi = \hat{\varphi}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\varphi}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \widehat{\varphi}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}) , \\ \boldsymbol{\Gamma}_{\boldsymbol{A}} = -[\nabla_{\boldsymbol{A}} \varphi] = \boldsymbol{0} , \end{cases}$$

and that all remaining stress-like coefficients $(\boldsymbol{\tau}_e, \boldsymbol{\tau}_d, \boldsymbol{\Gamma}_e, \boldsymbol{\Gamma}_\pi)$ obtained from energy derivatives are also functions of state exclusively. In light of the elastic-viscous stress decomposition (7.2), this appropriately restricts any deformation rate dependence of stress to the 'viscous' stress function $\boldsymbol{\tau}_f$.

• ***Purely dissipative inelastic mechanisms***. Many well-understood inelastic deformation mechanisms are purely dissipative in that they are responsible for the loss of mechanical energy while active, but have no internal energy associated with their present state. The frictional action of the dashpot in a (Maxwell-type) visco-elastic model consisting of a spring and dashpot connected in series is an example of such a mechanism. Clearly, any dislocation state variables associated with inelastic mechanisms of this type should not appear as arguments of the internal energy function.

• ***Non-viscous***. This material hypothesis is appropriate for materials whose *response* is determined as a continuous function of state (continuous stimuli elicit continuous response) requiring that both material response functions (energy & stress) be independent of the rates of deformation {cf. Dashner (1986b), Sec. 3B} comprising the set $\boldsymbol{A}$.

• ***Rate independent***. Rate independent materials are *non-viscous* materials for which all inelastic rate functions from the set

$$\{m_k\}_{k=1}^{N+2} = \left[\mathbf{D}_p, \mathbf{\Omega}_e, \boldsymbol{\pi} \equiv \{\pi_\alpha\}_{\alpha=1}^{N}\right] \tag{7.8}$$

are insensitive to a change of time scale. This condition takes the mathematical form

$$m_k(\mathbf{F}_e, \boldsymbol{p}, \boldsymbol{A}) = \tfrac{1}{\beta}\left[m_k(\mathbf{F}_e, \boldsymbol{p}, \mathbf{T}_\beta \boldsymbol{A})\right] \;;\quad k = 1, \dots, N+2, \tag{7.9}$$

with

$$\boldsymbol{A} = \{\mathbf{A}_k\}_{k=1}^{M} \quad \& \quad \mathbf{T}_\beta \boldsymbol{A} = \{\beta^k \mathbf{A}_k\}_{k=1}^{M}, \tag{7.10}$$

for arbitrary real-valued $\beta > 0$. For such materials, the *response* and *state* trajectories associated with a specified deformation program are uneffected by the elapsed time required to complete it.

• ***Elastically compliant***. Materials are said to be *elastically compliant* if, at each accessible state, there exists a non-trivial (spanning) subset of imposed material deformations which *would* proceed in a purely elastic fashion. This is commonly achieved through the introduction of yield criteria. Typically, this involves the introduction of a state dependent, scalar-valued, non-positive definite, *yield* function

$$\eta = \hat{\eta}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\eta}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \bar{\eta}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}) \le 0,$$

which defines a purely elastic state subspace $(\eta < 0)$, and restricts the activity of inelastic mechanisms through the elastic loading conditions

$$\left\{\begin{array}{c} \underline{\eta < 0} \\ \langle \text{or} \rangle \\ \underline{\eta = 0} \quad \& \quad \underline{\overset{e}{\eta} = [\dot{\eta}]_{\mathbf{D}_p = \mathbf{\Omega}_e = \boldsymbol{\pi} = 0} \le 0} \end{array}\right\} \Rightarrow \mathbf{D}_p = \mathbf{\Omega}_e = \boldsymbol{\pi} = 0. \tag{7.11}$$

Another common material property which facilitates elastic compliance is associated with inelastic mechanisms which "have a natural time," in the sense of Hill (1959). Considering the full set of inelastic rate functions (7.8), a particular inelastic mechanism is said to have a *natural time* if its corresponding rate function conforms to the fast rate elastic limit

$$\lim_{\beta \to \infty} \left\{\tfrac{1}{\beta}\left[m_k(\mathbf{F}_e, \boldsymbol{p}, \mathbf{T}_\beta \boldsymbol{A})\right]\right\} = 0,$$

expressed in terms of the time scale transformation (7.10). In essence, such inelastic mechanisms "cannot be hurried" and have no time to develop (evolve) during a rapidly imposed deformation. To illustrate, consider the inelastic dashpot mechanism in the previously cited visco-elastic Maxwell model.

It is important to note that this property, and the above discussed property (7.9) associated with *rate independent* materials, are contradictory and therefore mutually exclusive. Consequently, a *rate independent* solid cannot also be *elastically compliant* unless all of its inelastic mechanisms are constrained by explicit *yield* criteria of the type (7.11). In the remainder of this article, a material shall be described as being *elastically compliant* if all of its inelastic mechanisms which DO NOT have a *natural time* are constrained by yield criteria of the above type.

As to theoretical consequences, it is a simple matter to argue that it is necessary to require that the viscous stress function independently satisfy the constitutive inequality

$$\boldsymbol{\tau}_f \bullet \mathbf{D} \ge 0$$

for *elastically compliant* materials. Similar considerations lead to the conclusion that

$$\boldsymbol{\tau}_f = \boldsymbol{0} \tag{7.12}$$

for materials which are both *elastically compliant* and *non-viscous*.

• ***Invariant elastic properties***. This descriptor characterizes materials whose elastic stress response is unaltered by inelastic deformation mechanisms. Simply put, such materials reveal nothing of their (perhaps complex) deformation history through the subsequent performance of standard experiments which do not test the elastic limit. An unstressed element of such material, after appropriate reorientation, cannot be distinguished from a comparable virgin element through the performance of tests which do not excite inelastic mechanisms.

Despite the early claims by a number of 'Lagrandites' that such materials are easily modeled by Lagrangian constitutive relations of the reduced form

$$(\psi, \mathbf{T}) = \Re(\mathbf{E}_e) \;\; ; \quad \mathbf{E}_e \equiv \mathbf{E} - \mathbf{E}_p ,$$

expressed in terms of the difference of the total and plastic strains, this naive and simplistic model, in fact, bears no physical relation to the material property described[17]. Within the context of the present Eulerian theory, these physical properties are accurately modeled by requiring a separable energy function of the form

$$\varphi = \varphi_e + \varphi_p \;\; ; \quad \begin{cases} \varphi_e = \hat{\varphi}_e(\mathbf{F}_e) = \tilde{\varphi}_e(\mathbf{v}, \mathbf{R}_e) = \bar{\varphi}_e(\mathbf{a}, \mathbf{R}_e) , \\ \varphi_p = \varphi_p(\boldsymbol{p}) . \end{cases} \tag{7.13}$$

In addition to the above described effect, this uncoupling of elastic and inelastic mechanisms has an impact on the form of the various stress coefficients. As a consequence of applicable invariance criteria and established rate equations, it must clearly follow that

$$\dot{\varphi}_e = \boldsymbol{0} \;\; ; \quad \dot{\mathbf{F}}_e = \mathbf{W}\mathbf{F}_e \;\; ; \quad \dot{\mathbf{R}}_e = \mathbf{W}\mathbf{R}_e \;\; ; \quad \dot{\mathbf{v}} = \mathbf{W}\mathbf{v} - \mathbf{v}\mathbf{W} \;\; ; \quad \dot{\mathbf{a}} = \mathbf{W}\mathbf{a} - \mathbf{a}\mathbf{W} ,$$

during any ongoing rigid body motion. From this, it is easily established that

$$\boldsymbol{0} = \dot{\varphi}_e = \frac{\partial \hat{\varphi}_e}{\partial \mathbf{F}_e} \bullet \dot{\mathbf{F}}_e = \frac{\partial \hat{\varphi}}{\partial \mathbf{F}_e} \bullet (\mathbf{W}\mathbf{F}_e) = \left(\frac{\partial \hat{\varphi}}{\partial \mathbf{F}_e} \mathbf{F}_e^T \right) \bullet \mathbf{W} ,$$

$$\Rightarrow \quad \underline{\left[\frac{\partial \hat{\varphi}}{\partial \mathbf{F}_e} \mathbf{F}_e^T \right]_A \bullet \mathbf{W} = \boldsymbol{0}} ,$$

$$\boldsymbol{0} = \dot{\varphi}_e = \frac{\partial \tilde{\varphi}_e}{\partial \mathbf{v}} \bullet \dot{\mathbf{v}} + \frac{\partial \tilde{\varphi}_e}{\partial \mathbf{R}_e} \bullet \dot{\mathbf{R}}_e ,$$

$$\boldsymbol{0} = \frac{\partial \tilde{\varphi}}{\partial \mathbf{v}} \bullet (\mathbf{W}\mathbf{v} - \mathbf{v}\mathbf{W}) + \frac{\partial \tilde{\varphi}}{\partial \mathbf{R}_e} \bullet (\mathbf{W}\mathbf{R}_e)$$

$$\boldsymbol{0} = \left(\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}} \mathbf{v} - \mathbf{v} \frac{\partial \tilde{\varphi}}{\partial \mathbf{v}} \right) \bullet \mathbf{W} + \left(\frac{\partial \tilde{\varphi}}{\partial \mathbf{R}_e} \mathbf{R}_e^T \right) \bullet \mathbf{W} ,$$

$$\Rightarrow \quad \underline{\left\{ 2 \left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}} \mathbf{v} \right]_A + \left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{R}_e} \mathbf{R}_e^T \right]_A \right\} \bullet \mathbf{W} = \boldsymbol{0}} ,$$

and, in identical fashion

---

[17] This issue is addressed in (Dashner, 1979).

$$\Rightarrow \quad \left\{ 2\left[\frac{\partial \hat{\varphi}}{\partial \mathbf{a}}\mathbf{a}\right]_A + \left[\frac{\partial \hat{\varphi}}{\partial \mathbf{R}_c}\mathbf{R}_c^T\right]_A \right\} \bullet \mathbf{W} = \mathbf{0},$$

for arbitrary antisymmetric $\mathbf{W}$. Thus, it is assured that

$$\left[\frac{\partial \hat{\varphi}}{\partial \mathbf{F}_c}\mathbf{F}_c^T\right]_A = \mathbf{0} \quad \Rightarrow \quad \frac{\partial \hat{\varphi}}{\partial \mathbf{F}_c}\mathbf{F}_c^T = \left[\frac{\partial \hat{\varphi}}{\partial \mathbf{F}_c}\mathbf{F}_c^T\right]_S,$$

$$\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{R}_c}\mathbf{R}_c^T\right]_A = -2\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_A,$$

$$\left[\frac{\partial \hat{\varphi}}{\partial \mathbf{R}_c}\mathbf{R}_c^T\right]_A = -2\left[\frac{\partial \hat{\varphi}}{\partial \mathbf{a}}\mathbf{a}\right]_A,$$

for materials having invariant elastic properties. This, of course, allows for the simplification/alteration of the above listed general expressions for the stress coefficients $\boldsymbol{\tau}_e$, $\boldsymbol{\tau}_d$ and $\boldsymbol{\Gamma}_c$. In particular

$$\boldsymbol{\tau}_e = \frac{\partial \hat{\varphi}}{\partial \mathbf{F}_c}\mathbf{F}_c^T \quad \& \quad \boldsymbol{\tau}_d = \mathbf{R}_e\left[\mathbf{F}_c^T\frac{\partial \hat{\varphi}}{\partial \mathbf{F}_c}\right]_S\mathbf{R}_c^T,$$
$$\boldsymbol{\Gamma}_c = 2\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_A = 2\left[\frac{\partial \hat{\varphi}}{\partial \mathbf{a}}\mathbf{a}\right]_A. \tag{7.14}$$

• ***Structurally isotropic***. As discussed earlier in Sections 2 and 5, materials which have an isotropic characteristic reference cell are deemed to be *structurally isotropic*. Virtually all polycrystalline metals and alloys fall into this category. For such materials, anisotropy is not inherent, or *structural*, but *induced* through the complex path dependent action and interaction of various sorts of dislocation distributions. The phenomenological modeling of these mechanisms through the introduction of appropriate dislocation state variables is, of course, the real and ongoing challenge. The present effort is to develop a broad-based, theoretically sound, and physically insightful template for the construction and testing of such models.

As previously noted, the relevance of the cell orientation tensor $\mathbf{R}_c$ as a state descriptor vanishes for such materials. This guarantees that

$$\frac{\partial \tilde{\varphi}}{\partial \mathbf{R}_c} = \frac{\partial \hat{\varphi}}{\partial \mathbf{R}_c} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\Gamma}_c = \mathbf{0},$$

and also serves to justify the explicit substitutions (7.15)

$$\mathbf{R}_e = \mathbf{I} \; ; \quad \mathbf{F}_e = \mathbf{v} \; ; \quad \boldsymbol{\Omega}_c = \mathbf{0},$$

leading to the reduced forms

$$\boldsymbol{\tau}_e = \left\{\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_S + \underset{\sim}{\mathbf{W}}_v\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_A\right\} = \underset{\sim}{\mathbf{H}}_2\frac{\partial \hat{\varphi}}{\partial \mathbf{a}},$$
$$\boldsymbol{\tau}_d = \left\{\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_S - \underset{\sim}{\mathbf{W}}_v\left[\frac{\partial \tilde{\varphi}}{\partial \mathbf{v}}\mathbf{v}\right]_A\right\} = \underset{\sim}{\mathbf{H}}_1\frac{\partial \hat{\varphi}}{\partial \mathbf{a}}, \tag{7.16}$$

and

$$\dot{\gamma} = \boldsymbol{\tau}_f \bullet \mathbf{D} + \boldsymbol{\tau}_d \bullet \mathbf{D}_p + \boldsymbol{\Gamma}_\pi \bullet \boldsymbol{\pi} \geq 0 \; ; \quad [\boldsymbol{\Gamma}_c = \mathbf{0}].$$

Of particular interest is the class of *structurally isotropic* materials which <u>also</u> have *invariant elastic properties*. In view of (4.3), and the above simplifications (7.14), (7.15)and (7.16), it is easily verified that

$$\begin{aligned}
&\varphi = \tilde{\varphi}_e(\mathbf{v}) + \varphi_p(\boldsymbol{p}) = \widehat{\varphi}_c(\mathbf{a}) + \varphi_p(\boldsymbol{p}), \\
&\boldsymbol{\tau}_d = \boldsymbol{\tau}_e = \frac{\partial\tilde{\varphi}}{\partial\mathbf{v}}\mathbf{v} = \frac{\partial\widehat{\varphi}}{\partial\mathbf{a}}, \\
&\boldsymbol{\Gamma}_\pi = -\frac{\partial\varphi_p}{\partial\boldsymbol{p}} = -\left\{\frac{\partial\varphi_p}{\partial\mathbf{p}_\alpha}\right\}_{\alpha=1}^{N} \quad \& \quad \boldsymbol{\Gamma}_c = \boldsymbol{0},
\end{aligned} \tag{7.17}$$

for materials of this type.

• ***Small elastic strain***. In many applications, elastic strains rarely, if ever, exceed values of order 10^{-3}. Under such circumstances, approximate forms for the above relations may suffice. With reference to (A.27), recall that any tensor $\mathbf{X}$ has a unique decomposition of the form

$$\mathbf{X} = [\mathbf{X}]_{C(\mathbf{a})} + [\mathbf{X}]_{C^*(\mathbf{a})} \;;\quad \left\{\begin{array}{l} [\mathbf{X}]_{C(\mathbf{a})} \in C(\mathbf{a}), \\ [\mathbf{X}]_{C^*(\mathbf{a})} \in C^*(\mathbf{a}), \end{array}\right.$$

expressed in terms of orthogonal tensor components which belong, respectively, to the *commutative subspace* for the natural elastic strain,

$$C(\mathbf{a}) \equiv \{\mathbf{X}: \mathbf{aX} = \mathbf{Xa}\},$$

and its corresponding *outer space*

$$C^*(\mathbf{a}) = outer[C(\mathbf{a})] \equiv \{\mathbf{Y}: \mathbf{Y}\bullet\mathbf{X} = \mathbf{0}\ \forall\ \mathbf{X}\in C(\mathbf{a})\}.$$

In view of the relations (4.4) and (4.5) which apply to symmetric tensors $\mathbf{X}=\mathbf{X}^T$, it is a relatively simple matter to verify the inequalities

$$\left.\begin{array}{l} \left|\underset{\sim}{\mathbf{H}}_1\mathbf{X} - \mathbf{X}\right| \le \frac{1}{3}\mu_{max}^2 \left|[\mathbf{X}]_{C^*(\mathbf{a})}\right| \\ \left|\underset{\sim}{\mathbf{H}}_2\mathbf{X} - \mathbf{X}\right| \le \frac{1}{6}\mu_{max}^2 \left|[\mathbf{X}]_{C^*(\mathbf{a})}\right| \end{array}\right\} ;\quad \mu_{max} \equiv max(|\mu_1|, |\mu_2|, |\mu_3|).$$

After verifying the bounding inequalities

$$\tfrac{1}{2}\mu_{max}^2 \le |\mathbf{a}'|^2 \le \tfrac{2}{3}\mu_{max}^2 \;;\quad \mathbf{a}' = dev(\mathbf{a}) \equiv \mathbf{a} - \tfrac{1}{3}tr(\mathbf{a})\mathbf{I},$$

involving the natural strain *deviator*, it then follows that

$$\left.\begin{array}{l} \left|\underset{\sim}{\mathbf{H}}_1\mathbf{X} - \mathbf{X}\right| \le \frac{2}{3}|\mathbf{a}'|^2|\mathbf{X}^*| \le \frac{2}{3}|\mathbf{a}'|^2|\mathbf{X}| \\ \left|\underset{\sim}{\mathbf{H}}_2\mathbf{X} - \mathbf{X}\right| \le \frac{1}{3}|\mathbf{a}'|^2|\mathbf{X}^*| \le \frac{1}{3}|\mathbf{a}'|^2|\mathbf{X}| \end{array}\right\} ;\quad \mathbf{X}^* = [\mathbf{X}]_{C^*(\mathbf{a})} \in C^*(\mathbf{a}).$$

These inequalities not only serve to reinforce the already stated fact that

$$\left.\begin{array}{c} \mathbf{a} = a\mathbf{I} \\ \langle\text{or}\rangle \\ \mathbf{X}\in C(\mathbf{a}) \end{array}\right\} \Rightarrow \underset{\sim}{\mathbf{H}}_k\mathbf{X} = \mathbf{X} \;;\quad k = 1, 2,$$

but also establish the small strain expansions[18]

[18] Note that for the cited circumstance for which $|\mathbf{a}| \le 10^{-3}$, this approximation can be expected to produce six (6) significant figure accuracy.

$$\underset{\sim}{\mathbf{H}}_k\mathbf{X} = \mathbf{X} + \text{Order}\{\|\mathbf{a}'\|^2\|\mathbf{X}\|\} \ ; \quad k=1,2 ,$$

and the associated approximation

$$|\mathbf{a}'| \le |\mathbf{a}| << 1 \quad \Rightarrow \quad \underset{\sim}{\mathbf{H}}_k\mathbf{X} \approx \mathbf{X} \ ; \quad k=1,2 .$$

With this, examination of the general forms (7.5) for the various stress coefficients leads to the final conclusion that

$$\boldsymbol{\tau}_d \approx \boldsymbol{\tau}_c \approx \frac{\partial\widehat{\varphi}}{\partial\mathbf{a}} \approx \left[\frac{\partial\widetilde{\varphi}}{\partial\mathbf{v}}\mathbf{v}\right]_S \approx \frac{1}{2}\left\{\left[\frac{\partial\hat{\varphi}}{\partial\mathbf{F}_e}\mathbf{F}_e^T\right]_S + \mathbf{R}_c\left[\mathbf{F}_c^T\frac{\partial\hat{\varphi}}{\partial\mathbf{F}_c}\right]_S\mathbf{R}_c^T\right\}$$

whenever this approximation is deemed to produce results of sufficient accuracy.

8. Elastic recycling

This paper shall also detail the implications of a concept of material stability pertaining to the work done and energy released during closed deformation cycles. Such analysis is relatively straightforward within the classical Lagrangian theoretical structure. Here, the absence of an explicit measure of total deformation presents an interesting, but not insurmountable challenge.

As in (5.4), consider a smooth process of material deformation

$$\mathbf{F} = \mathbf{F}(t) \ ; \ t \ge t_i \ \langle\text{with}\rangle \ \mathbf{F}(t_i) = \mathbf{I} ,$$

imposed on a material element from a base ($t=t_i$) configuration in which its state is characterized by the initial values

$$\mathbf{F}_e(t_i) = \mathbf{F}_{e_o} \ ; \ \mathbf{v}(t_i) = \mathbf{v}_o \ ; \ \mathbf{R}_e(t_i) = \mathbf{R}_{e_o} \ ; \ \boldsymbol{p}(t_i) = \boldsymbol{p}_o ,$$
$$\mathbf{b}(t_i) = \mathbf{b}_o \ ; \ \mathbf{c}(t_i) = \mathbf{c}_o \ ; \ \mathbf{u}(t_i) = \mathbf{u}_o \ ; \ \mathbf{a}(t_i) = \mathbf{a}_o .$$

During such a process, this material element's configuration and state evolve according to the above established rate relations. At each instant during this process, consider the proper, non-singular 'deformation gradient' defined by the expression

$$\mathbf{F}_r(t) \equiv \left[\mathbf{v}(t)\mathbf{F}^{-T}(t)\mathbf{F}^{-1}(t)\mathbf{v}(t)\right]^{1/2}\mathbf{u}(t) \ ; \quad t \ge t_i . \tag{8.1}$$

This so-called 'elastic recovery' tensor has certain easily established properties. First of all, through the algebraic development

$$\begin{aligned}
[\mathbf{F}_r(t)\mathbf{F}(t)]^T[\mathbf{F}_r(t)\mathbf{F}(t)] &= \mathbf{F}^T(t)\left[\mathbf{F}_r^T(t)\mathbf{F}_r(t)\right]\mathbf{F}(t) \\
(\mathbf{F}_r\mathbf{F})^T(\mathbf{F}_r\mathbf{F}) &= \mathbf{F}^T\left\{\left[\mathbf{u}\left(\mathbf{v}\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{v}\right)^{1/2}\right]\left[\left(\mathbf{v}\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{v}\right)^{1/2}\mathbf{u}\right]\right\}\mathbf{F} \\
&= \mathbf{F}^T\left[\mathbf{u}\left(\mathbf{v}\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{v}\right)\mathbf{u}\right]\mathbf{F} \\
(\mathbf{F}_r\mathbf{F})^T(\mathbf{F}_r\mathbf{F}) &= \mathbf{I} ,
\end{aligned}$$

$$\Rightarrow \ \mathbf{F}_r(t)\mathbf{F}(t) = \mathbf{Q}_r(t) \ \ \langle\text{where}\rangle \ \ \mathbf{Q}_r\mathbf{Q}_r^T = \mathbf{Q}_r^T\mathbf{Q}_r = \mathbf{I} , \tag{8.2}$$

it is established that if this deformation were to be superposed onto the current deformation $\mathbf{F}(t)$, the net effect would be nothing more than a rigid rotation. Thus, **superposition of the additional deformation $\mathbf{F}_r(t)$ onto the current deformation $\mathbf{F}(t)$ would create a *closed deformation cycle*.** Next, the development

$$\mathbf{F}_r(t) = \left[\mathbf{v}(t)\mathbf{F}^{-T}(t)\mathbf{F}^{-1}(t)\mathbf{v}(t)\right]^{1/2}\mathbf{u}(t)$$
$$\mathbf{F}_r(t) \overset{F(t)=Q}{=} \left[\mathbf{v}(t)\mathbf{Q}\mathbf{Q}^T\mathbf{v}(t)\right]^{1/2}\mathbf{u}(t) \;\; ; \;\; \left[\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\right] \tag{8.3}$$
$$\mathbf{F}_r(t) \overset{F(t)=Q}{=} \sqrt{\mathbf{v}^2(t)}\,\mathbf{u}(t) = \mathbf{v}(t)\mathbf{u}(t) = \mathbf{I}\,,$$

justifies the conclusion that **the elastic recovery tensor reduces to the identity mapping whenever the current deformation F(t) already describes a closed deformation cycle**. The final observation is simply that **the product combination**

$$\mathbf{F}_r(t)\mathbf{v}(t) = \left[\mathbf{v}(t)\mathbf{F}^{-T}(t)\mathbf{F}^{-1}(t)\mathbf{v}(t)\right]^{1/2} = \left[\mathbf{F}_r(t)\mathbf{v}(t)\right]^T \tag{8.4}$$

is symmetric.

The *elastic recovery tensor* (8.1) is now used to define a subsidiary elastic cell placement tensor

$$\mathbf{F}_{e_r}(t) \equiv \mathbf{F}_r(t)\mathbf{F}_e(t) \;\; ; \;\; t \geq t_i\,,$$

and its related symmetric and orthogonal constituents

$$\left.\begin{aligned}
\mathbf{v}_r(t) &= \sqrt{\mathbf{F}_{e_r}\mathbf{F}_{e_r}^T} \\
\mathbf{u}_r(t) &= \mathbf{v}_r^{-1} = \sqrt{\mathbf{F}_{e_r}^{-T}\mathbf{F}_{e_r}^{-1}} \\
\mathbf{R}_{e_r}(t) &= \mathbf{u}_r\mathbf{F}_{e_r} \;\Leftrightarrow\; \mathbf{F}_{e_r} = \mathbf{v}_r\mathbf{R}_{e_r} \\
\mathbf{b}_r(t) &= \mathbf{F}_{e_r}\mathbf{F}_{e_r}^T = \mathbf{F}_r\mathbf{b}\mathbf{F}_r^T = \mathbf{v}_r^2 \\
\mathbf{c}_r(t) &= \mathbf{F}_{e_r}^{-T}\mathbf{F}_{e_r}^{-1} = \mathbf{F}_r^{-T}\mathbf{c}\mathbf{F}_r^{-1} = \mathbf{u}_r^2 \\
\mathbf{a}_r(t) &= ln(\mathbf{v}_r)
\end{aligned}\right\} \;\; ; \;\; t \geq t_i\,. \tag{8.5}$$ [19]

With reference to (5.1), it is immediately clear that **these new variables represent the updated values of the current state descriptors $\left[\mathbf{F}_e, \mathbf{v}, \mathbf{u}, \mathbf{R}_e, \mathbf{b}, \mathbf{c}, \mathbf{a}\right]_{t \geq t_i}$ after the superposition of a (hypothetical) purely elastic deformation $\mathbf{F}_r$**. Thus, the designation of $\mathbf{F}_r$ as the *elastic recovery tensor* and of $\left[\mathbf{F}_{e_r}, \mathbf{v}_r, \mathbf{u}_r, \mathbf{R}_{e_r}, \mathbf{b}_r, \mathbf{c}_r, \mathbf{a}_r\right]_{t \geq t_i}$ as the instantaneous *elastic recovery variables*. It is important to note that, because of (8.3), **the elements of this elastic recovery set are identical to the actual elastic state variables whenever the deformation cycle is *closed*, that is, represents nothing more than a rigid rotation**. This is formally expressed as

$$\left[\mathbf{F}_{e_r}, \mathbf{v}_r, \mathbf{u}_r, \mathbf{R}_{e_r}, \mathbf{b}_r, \mathbf{c}_r, \mathbf{a}_r\right]_{t \geq t_i} \overset{F(t)=Q}{=} \left[\mathbf{F}_e, \mathbf{v}, \mathbf{u}, \mathbf{R}_e, \mathbf{b}, \mathbf{c}, \mathbf{a}\right]_{t \geq t_i}\,.$$

Another important observation follows from the rearrangement

$$\mathbf{F}_{e_r}(t) = \mathbf{F}_r(t)\mathbf{F}_e(t) \;\; ; \;\; \left[\mathbf{F}_{e_r} = \mathbf{v}_r\mathbf{R}_{e_r} \;\&\; \mathbf{F}_e = \mathbf{v}\mathbf{R}_e\right],$$
$$\mathbf{v}_r\mathbf{R}_{e_r} = \mathbf{F}_r(\mathbf{v}\mathbf{R}_e) = (\mathbf{F}_r\mathbf{v})\mathbf{R}_e\,,$$

and the previously noted symmetry property (8.4). After recalling the classical result pertaining to the uniqueness of the polar decomposition, it immediately follows that

$$\mathbf{v}_r(t) = \mathbf{F}_r(t)\,\mathbf{v}(t)\,,$$
$$\mathbf{R}_{e_r}(t) = \mathbf{R}_e(t)\,,$$

[19] With reference to (2.11), (3.2), and (4.1), it is apparent that these relationships exactly mirror those for the actual (non-subscripted) set of elastic variables.

This allows for the following refinement

$$\begin{aligned}
&\mathbf{F}_r \equiv \left[\mathbf{v}\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{v}\right]^{1/2}\mathbf{u}\,, \\
&\mathbf{F}_{e_r} = \mathbf{F}_r\mathbf{F}_e = \mathbf{v}_r\mathbf{R}_{e_r}\,, \\
&\mathbf{R}_{e_r} = \mathbf{R}_e\,, \\
&\mathbf{v}_r = \mathbf{F}_r\mathbf{v} = \mathbf{v}\,\mathbf{F}_r^T = \mathbf{v}_r^T\,, \\
&\mathbf{u}_r = \mathbf{u}\mathbf{F}_r^{-1} = \mathbf{F}_r^{-T}\mathbf{u} = \mathbf{u}_r^T = \mathbf{v}_r^{-1}\,, \\
&\mathbf{b}_r = \mathbf{v}_r^2 = \mathbf{v}_r\mathbf{v}_r^T = \mathbf{F}_r\mathbf{b}\mathbf{F}_r^T\,, \\
&\mathbf{c}_r = \mathbf{u}_r^2 = \mathbf{u}_r^T\mathbf{u}_r = \mathbf{F}_r^{-T}\mathbf{c}\mathbf{F}_r^{-1} = \mathbf{b}_r^{-1}\,, \\
&\mathbf{a}_r = ln(\mathbf{v}_r)\,,
\end{aligned} \tag{8.6}$$

of the above relations (8.5). Upon comparison of $(8.6)_4$ with $(5.5)_3$, it is further revealed that **the elastic recovery tensor is the deformation gradient that *would be measured* by shadow frame observers during a purely elastic deformation process (superposed onto the existing deformation) which *closes* the deformation cycle**. Since it has already been established that the semi-Lagrangian inelastic state variables

$$\boldsymbol{p} \equiv \{\mathbf{p}_\alpha\}_{\alpha=1}^{N}\ ;\quad \mathbf{p}_\alpha = {}_\alpha\mathbf{T}_u(\mathbf{q}_\alpha) \quad\Leftrightarrow\quad \mathbf{q}_\alpha = {}_\alpha\mathbf{T}_v(\mathbf{p}_\alpha)$$

are observed to remain constant by shadow frame observers during any such process, it makes sense to form a complete set of elastic recovery variables through the final definition

$$\boldsymbol{p}_r(t) \equiv \boldsymbol{p}(t)\ ;\quad t \ge t_i\,.$$

In general, the subscript "r" shall be affixed to any state dependent quantity or function whenever it is to be evaluated at this so-called *elastically recovered state* rather than at the actual or *true state*. For example,

$$\left[\mathbf{F}_e, \mathbf{v}, \mathbf{u}, \mathbf{b}, \mathbf{c}, \mathbf{a}, \mathbf{R}_e, \boldsymbol{p}\right]_r = \left[\mathbf{F}_{e_r}, \mathbf{v}_r, \mathbf{u}_r, \mathbf{b}_r, \mathbf{c}_r, \mathbf{a}_r, \mathbf{R}_{e_r}, \boldsymbol{p}_r\right]$$

or, if

$$\Psi = \hat{\Psi}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\Psi}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \widehat{\Psi}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p})\,,$$

then

$$\Psi_r = \hat{\Psi}(\mathbf{F}_{e_r}, \boldsymbol{p}_r) = \tilde{\Psi}(\mathbf{v}_r, \mathbf{R}_{e_r}, \boldsymbol{p}_r) = \widehat{\Psi}(\mathbf{a}_r, \mathbf{R}_{e_r}, \boldsymbol{p}_r)\,.$$

Before proceeding, it is worthwhile to reiterate that **this set of *elastic recovery variables***

(i) is well defined at any instant during a continuing deformation process proceeding from some preselected base state;

(ii) are identical to the true state variables at any instant for which this deformation reduces to a rigid body rotation, that is $\mathbf{F}\mathbf{F}^T = \mathbf{I}$;

(iii) generally differ from the set of true state variables only through the symmetric elastic deformation measures since, in general

$$\left.\begin{aligned} \mathbf{R}_{e_r}(t) &= \mathbf{R}_e(t) \\ \boldsymbol{p}_r(t) &= \boldsymbol{p}(t) \end{aligned}\right\} \Leftrightarrow \left[\mathbf{R}_e, \boldsymbol{p}\right]_r = \left[\mathbf{R}_e, \boldsymbol{p}\right].$$

To complete this section, explicit rate equations for each of these elastic recovery variables shall be derived. In view of the established rate relations (3.26), (5.8), and the above observation (**iii**), it is immediately clear that

$$\overset{s}{\mathbf{R}}_{e_r} = \overset{s}{\mathbf{R}}_e = \mathbf{\Omega}_e\mathbf{R}_e = \mathbf{\Omega}_e\mathbf{R}_{e_r},$$
$$\overset{s}{\boldsymbol{p}}_r = \overset{s}{\boldsymbol{p}} = \boldsymbol{\pi},$$

in which the elastic spin $\mathbf{\Omega}_e$ and inelastic rates $\boldsymbol{\pi}$ are determined as functions of the true state as detailed in Section 5. The time rates for the symmetric elastic deformation measures can be determined by exploiting the relations (8.2) and (8.6). Beginning with the expansion

$$\begin{aligned}
\mathbf{c}_r(t) &= \mathbf{u}_r^2(t)\\
\mathbf{c}_r &= \mathbf{u}_r\left[\mathbf{Q}_r\mathbf{Q}_r^T\right]\mathbf{u}_r \;; \quad \left[\mathbf{Q}_r\mathbf{Q}_r^T = \mathbf{I}\right]\\
&= \mathbf{u}_r\left[(\mathbf{F}_r\mathbf{F})(\mathbf{F}_r\mathbf{F})^T\right]\mathbf{u}_r \;; \quad \left[\mathbf{F}_r\mathbf{F} = \mathbf{Q}_r\right]\\
&= (\mathbf{u}_r\mathbf{F}_r)(\mathbf{F}\mathbf{F}^T)(\mathbf{u}_r\mathbf{F}_r)^T\\
&= \mathbf{u}(\mathbf{F}\mathbf{F}^T)\mathbf{u} \;; \quad \left[\mathbf{u}_r = \mathbf{u}\mathbf{F}_r^{-1} \Rightarrow \mathbf{u}_r\mathbf{F}_r = \mathbf{u}\right]\\
\mathbf{c}_r(t) &= \left[\mathbf{u}(t)\mathbf{F}(t)\right]\left[\mathbf{u}(t)\mathbf{F}(t)\right]^T,
\end{aligned}$$

it follows that

$$\dot{\mathbf{c}}_r = (\dot{\overline{\mathbf{uF}}})(\mathbf{uF})^T + (\mathbf{uF})(\dot{\overline{\mathbf{uF}}})^T \;; \quad \left\{ \begin{aligned} \dot{\overline{\mathbf{uF}}} &= \dot{\mathbf{u}}\mathbf{F} + \mathbf{u}\dot{\mathbf{F}}\\ &= (\mathbf{L}_s\mathbf{u} - \mathbf{uL})\mathbf{F} + \mathbf{u}(\mathbf{LF})\\ \dot{\overline{\mathbf{uF}}} &= \mathbf{L}_s(\mathbf{uF}), \end{aligned} \right.$$

$$\begin{aligned}
\dot{\mathbf{c}}_r &= \left[\mathbf{L}_s(\mathbf{uF})\right](\mathbf{uF})^T + (\mathbf{uF})\left[\mathbf{L}_s(\mathbf{uF})\right]^T\\
&= \mathbf{L}_s\left[(\mathbf{uF})(\mathbf{uF})^T\right] + \left[(\mathbf{uF})(\mathbf{uF})^T\right]\mathbf{L}_s^T\\
\dot{\mathbf{c}}_r &= \mathbf{L}_s\mathbf{c}_r + \mathbf{c}_r\mathbf{L}_s^T .
\end{aligned}$$

In view of the flow rate relations (3.24), the corresponding shadow rate evolution equation

$$\overset{s}{\mathbf{c}}_r = \mathbf{c}_r\mathbf{D}_p + \mathbf{D}_p\mathbf{c}_r \tag{8.7}$$

is immediately established. In this expression, the plastic (shadow flow) deformation rate $\mathbf{D}_p$ is determined as a function of the true state as noted above.

The remaining rate expressions are quickly realized after collecting the previously established results (2.14), (3.5), (3.6), (3.13) and (4.2), *viz.*

$$\begin{aligned}
\overset{o}{\mathbf{c}} &= -\mathbf{cD} - \mathbf{Dc} + 2\mathbf{u}\mathbf{D}_p\mathbf{u},\\
\overset{o}{\mathbf{b}} &= \mathbf{bD} + \mathbf{Db} - 2\mathbf{v}\mathbf{D}_p\mathbf{v},\\
\overset{o}{\mathbf{u}} &= \left\{ \begin{aligned} &(\mathbf{D}_p + \mathbf{\Omega}_s)\mathbf{u} - \mathbf{uD}\\ &\mathbf{u}(\mathbf{D}_p - \mathbf{\Omega}_s) - \mathbf{Du} \end{aligned} \right.\\
\overset{o}{\mathbf{v}} &= \left\{ \begin{aligned} &\mathbf{Dv} - \mathbf{v}(\mathbf{D}_p + \mathbf{\Omega}_s)\\ &\mathbf{vD} - (\mathbf{D}_p - \mathbf{\Omega}_s)\mathbf{v} \end{aligned} \right.
\end{aligned} \;\; ; \quad \mathbf{\Omega}_s = \underset{\sim}{\mathbf{W}}_u \cdot (\mathbf{D} + \mathbf{D}_p) = -\underset{\sim}{\mathbf{W}}_v \cdot (\mathbf{D} + \mathbf{D}_p), \tag{8.8}$$
$$\overset{o}{\mathbf{a}} = \underset{\sim}{\mathbf{H}}_1 \cdot \mathbf{D} - \underset{\sim}{\mathbf{H}}_2 \cdot \mathbf{D}_p,$$

for the corotational rates of the (true) elastic deformation measures. Realizing that all of these tensors are uniquely determined by specification of any one, it likewise follows that the rate equation for any one determines the rate equations for all. Direct comparison of (8.7) with $(8.8)_1$ reveals that the desired shadow rate expressions for the elastic recovery variables will follow from the explicit substitutions

$$\begin{aligned}
[\mathbf{v},\mathbf{u},\mathbf{b},\mathbf{c},\mathbf{a}] &\rightarrow [\mathbf{v},\mathbf{u},\mathbf{b},\mathbf{c},\mathbf{a}]_r \\
\overset{o}{(\ )} &\rightarrow \overset{s}{(\ )}_r \\
\mathbf{D}_p &\rightarrow \boldsymbol{0} \\
\mathbf{D} &\rightarrow -\mathbf{D}_p
\end{aligned}$$

into the above corotational rate expressions. This is easily accomplished, resulting in the forms

$$\begin{aligned}
&\overset{s}{\mathbf{c}}_r = \mathbf{c}_r\mathbf{D}_p + \mathbf{D}_p\mathbf{c}_r\,, \\
&\overset{s}{\mathbf{b}}_r = -\mathbf{b}_r\mathbf{D}_p - \mathbf{D}_p\mathbf{b}_r\,, \\
&\overset{s}{\mathbf{u}}_r = \begin{cases} \boldsymbol{\Omega}_{s_r}\mathbf{u}_r + \mathbf{u}_r\mathbf{D}_p \\ -\mathbf{u}_r\boldsymbol{\Omega}_{s_r} + \mathbf{D}_p\mathbf{u}_r \end{cases} \\
&\overset{s}{\mathbf{v}}_r = \begin{cases} -\mathbf{D}_p\mathbf{v}_r - \mathbf{v}_r\boldsymbol{\Omega}_{s_r} \\ -\mathbf{v}_r\mathbf{D}_p + \boldsymbol{\Omega}_{s_r}\mathbf{v}_r \end{cases} \Bigg\rangle\ ; \quad \boldsymbol{\Omega}_{s_r} = -\underset{\sim}{\mathbf{W}}_{u_r}\cdot\mathbf{D}_p = \underset{\sim}{\mathbf{W}}_{v_r}\cdot\mathbf{D}_p\,, \\
&\overset{s}{\mathbf{a}}_r = -\underset{\sim}{\mathbf{H}}_{1_r}\cdot\mathbf{D}_p\,,
\end{aligned}$$

and, for completeness (8.9)

$$\begin{aligned}
&\mathbf{F}_{e_r} = \mathbf{v}_r\mathbf{R}_e\ ; \quad [\mathbf{R}_{e_r}=\mathbf{R}_e]\,, \\
&\overset{s}{\mathbf{F}}_{e_r} = \overset{s}{\mathbf{v}}_r\mathbf{R}_e + \mathbf{v}_r\overset{s}{\mathbf{R}}_e \\
&\quad = \left(-\mathbf{v}_r\mathbf{D}_p + \boldsymbol{\Omega}_{s_r}\mathbf{v}_r\right)\mathbf{R}_e + \mathbf{v}_r\left(\boldsymbol{\Omega}_e\mathbf{R}_e\right)\ ; \quad [\mathbf{v}_r = \mathbf{F}_{e_r}\mathbf{R}_e^T] \\
&\overset{s}{\mathbf{F}}_{e_r} = -\mathbf{F}_{e_r}\left[\mathbf{R}_e^T\left(\mathbf{D}_p - \boldsymbol{\Omega}_e\right)\mathbf{R}_e\right] + \boldsymbol{\Omega}_{s_r}\mathbf{F}_{e_r}\,.
\end{aligned}$$

As repeatedly emphasized, the relations (3.27) guarantee that the plastic stretching rate $\mathbf{D}_p$ and reference cell spin $\boldsymbol{\Omega}_e$ vanish during purely elastic deformations. Consequently, a fourth key property, namely

(iv) each and every one of the *elastic recovery variables*

$$[\mathbf{F}_e,\mathbf{v},\mathbf{u},\mathbf{b},\mathbf{c},\mathbf{a},\mathbf{R}_e,\boldsymbol{p}]_r = [\mathbf{F}_{e_r},\mathbf{v}_r,\mathbf{u}_r,\mathbf{b}_r,\mathbf{c}_r,\mathbf{a}_r,\mathbf{R}_{e_r},\boldsymbol{p}_r]$$

is perceived to remain <u>constant</u> by shadow frame observers during any purely elastic deformation process,

may now be appended to the above observations (**i**) - (**iii**). To underscore the significance of this, the above rate relations are now recompiled and displayed in terms of their respective *elastic* and *non-elastic* shadow rate components (6.2), *viz*

$$\overset{s}{\boldsymbol{p}}_r = \overset{e}{\boldsymbol{p}}_r + \overset{n}{\boldsymbol{p}}_r\ ; \quad \begin{cases} \overset{e}{\boldsymbol{p}}_r = \boldsymbol{0} \\ \overset{n}{\boldsymbol{p}}_r = \boldsymbol{\pi} \end{cases} \Big\rangle\ ; \quad [\boldsymbol{p}_r = \boldsymbol{p}]\,,$$

$$\overset{s}{\mathbf{R}}_{e_r} = \overset{e}{\mathbf{R}}_{e_r} + \overset{n}{\mathbf{R}}_{e_r}\ ; \quad \begin{cases} \overset{e}{\mathbf{R}}_{e_r} = \boldsymbol{0} \\ \overset{n}{\mathbf{R}}_{e_r} = \boldsymbol{\Omega}_e\mathbf{R}_{e_r} \end{cases} \Big\rangle\ ; \quad [\mathbf{R}_{e_r} = \mathbf{R}_e]\,,$$

$$\overset{s}{\mathbf{v}}_r = \overset{e}{\mathbf{v}}_r + \overset{n}{\mathbf{v}}_r\ ; \quad \begin{cases} \overset{e}{\mathbf{v}}_r = \boldsymbol{0}\,, \\ \overset{n}{\mathbf{v}}_r = \left(\underset{\sim}{\mathbf{W}}_{v_r}\cdot\mathbf{D}_p\right)\mathbf{v}_r - \mathbf{v}_r\mathbf{D}_p = -\left[\mathbf{v}_r\left(\underset{\sim}{\mathbf{W}}_{v_r}\cdot\mathbf{D}_p\right) + \mathbf{D}_p\mathbf{v}_r\right], \end{cases}$$

$$\overset{s}{\mathbf{F}}_{e_r} = \overset{e}{\mathbf{F}}_{e_r} + \overset{n}{\mathbf{F}}_{e_r} \;\; ; \;\; \begin{cases} \overset{e}{\mathbf{F}}_{e_r} = \mathbf{0} \,, \\ \overset{n}{\mathbf{F}}_{e_r} = \left(\underset{\sim}{\mathbf{W}}_{v_r} \mathbf{D}_p\right)\mathbf{F}_{e_r} - \mathbf{F}_{e_r}\left[\mathbf{R}^T_{e_r}(\mathbf{D}_p - \mathbf{\Omega}_e)\mathbf{R}_{e_r}\right] \;\; ; \;\; [\mathbf{R}_{e_r} = \mathbf{R}_e] \,, \end{cases}$$

$$\overset{s}{\mathbf{a}}_r = \overset{e}{\mathbf{a}}_r + \overset{n}{\mathbf{a}}_r \;\; ; \;\; \begin{cases} \overset{e}{\mathbf{a}}_r = \mathbf{0} \,, \\ \overset{n}{\mathbf{a}}_r = -\underset{\sim}{\mathbf{H}}_{1_r} \mathbf{D}_p \,. \end{cases}$$

Direct comparison of these with the corresponding rate relations (6.3) for the true state variables leads to some important conclusions. Taking particular note of the identical forms for the *non-elastic* rates, and of the vanishing *elastic* rates, it follows from the above observation (**ii**), and the shadow rate decomposition (6.4) for scalar or tensor-valued state functions (6.1), that

$$\overset{s}{\Psi}_r \overset{F(t)=Q}{=} \overset{n}{\Psi} = \overset{s}{\Psi} - \overset{e}{\Psi} \tag{8.10}$$

at any instant for which the measured deformation amounts to a rigid rotation, *i.e.* $\mathbf{F}\mathbf{F}^T = \mathbf{I}$.

This comparison also facilitates an immediate conclusion as to the proper form for the shadow frame time derivative of a scalar-valued, frame invariant state function evaluated at the corresponding elastically recovered state

$$\mathsf{f}_r = \begin{cases} \hat{\mathsf{f}}(\mathbf{F}_{e_r}, \boldsymbol{p}_r) = \hat{\mathsf{f}}(\mathbf{F}_{e_r}, \boldsymbol{p}) \,, \\ \tilde{\mathsf{f}}(\mathbf{v}_r, \mathbf{R}_{e_r}, \boldsymbol{p}_r) = \tilde{\mathsf{f}}(\mathbf{v}_r, \mathbf{R}_e, \boldsymbol{p}) \,, \\ \overset{\frown}{\mathsf{f}}(\mathbf{a}_r, \mathbf{R}_{e_r}, \boldsymbol{p}_r) = \overset{\frown}{\mathsf{f}}(\mathbf{a}_r, \mathbf{R}_e, \boldsymbol{p}) \,. \end{cases}$$

Specifically, after reviewing the detailed development leading to (6.5), (6.6) and (6.7), it is revealed that the desired rate expression can be concisely expressed as

$$\boxed{\dot{\mathsf{f}}_r = \left[\nabla_{D_p}\mathsf{f}\right]_r \bullet \mathbf{D}_p + \left[\nabla_{\Omega_e}\mathsf{f}\right]_r \bullet \mathbf{\Omega}_e + \left[\nabla_{\pi}\mathsf{f}\right]_r \bullet \boldsymbol{\pi}} \,,$$

$$\dot{\mathsf{f}}_r = \overset{s}{\mathsf{f}}_r = \overset{e}{\mathsf{f}}_r + \overset{n}{\mathsf{f}}_r \;\; ; \;\; \begin{cases} \overset{e}{\mathsf{f}}_r = 0 \,, \\ \overset{n}{\mathsf{f}}_r = \left[\nabla_{D_p}\mathsf{f}\right]_r \bullet \mathbf{D}_p + \left[\nabla_{\Omega_e}\mathsf{f}\right]_r \bullet \mathbf{\Omega}_e + \left[\nabla_{\pi}\mathsf{f}\right]_r \bullet \boldsymbol{\pi} \,. \end{cases}$$

Here, it is critical to note that the tensor 'gradient' coefficient functions (6.6) are evaluated at the instantaneous elastically recovered state rather than at the true state (as indicated by the "r" subscript), while the inelastic rates $\left[\mathbf{D}_p, \mathbf{\Omega}_e, \boldsymbol{\pi}\right]$ are determined as functions of the true state as previously noted.

As a final exercise, and useful preliminary for the stability considerations to follow, this last result shall now be used to consider the change in such a scalar-valued state function realized during a closed deformation cycle initiated at $t = t_i$ and concluding at $t = t_f > t_i$. Letting

$$\mathbf{F} = \mathbf{F}(t) \;\; ; \;\; t \geq t_i$$

measure the material deformation relative to the base state at $t = t_i$, it is clear that

$$\mathbf{F}(t_i) = \mathbf{I} \quad \& \quad \mathbf{F}(t_f) = \mathbf{Q}_f$$

for some orthogonal rotation tensor $\mathbf{Q}_f$. In terms of the set of elastic recovery variables defined relative to this base state, it follows from property (**ii**), and the above derived result, that

$$\Delta f = f(t_f) - f(t_i) = f_r(t_f) - f_r(t_i) = \int_{t_i}^{t_f} \dot{f}_r \, dt \,,$$

$$\boxed{\Delta f = \int_{t_i}^{t_f} \left\{ \left[\nabla_{D_p} f\right]_r \bullet \mathbf{D}_p + \left[\nabla_{\Omega_e} f\right]_r \bullet \mathbf{\Omega}_e + \left[\nabla_{\pi} f\right]_r \bullet \boldsymbol{\pi} \right\} dt} \,. \tag{8.11}$$

Finally, after noting that the integrand vanishes during any purely elastic portion of the deformation process, it follows that the integral need only be evaluated over periods during which inelastic mechanisms are active.

9. Il'iushin stability

In addition to the restrictions imposed by the response and dissipation forms (7.5) and (7.7), it is common to require that materials also be stable in the sense of Il'iushin (1961), that is, that mechanical energy cannot be extracted from a material element during a closed deformation cycle. More specifically, this requires that, for such cycles, the energy dissipated must equal or exceed the internal energy released. This takes the explicit mathematical form

$$\Delta\gamma \equiv \int_{t_i}^{t_f} \dot{\gamma} \, dt \geq -(\Delta\varphi)$$

for any closed deformation cycle commencing at $t = t_i$ and terminating at $t = t_f$.

Upon substitution of the dissipation inequality (7.1), this requirement is seen to take the alternative form

$$\int_{t_i}^{t_f} \boldsymbol{\tau} \bullet \mathbf{D} \, dt = \Delta\gamma + \Delta\varphi \geq 0 \tag{9.1}$$

expressing the closed-cycle non-negativity of the 'stress work' per unit reference volume. In view of (7.7), and the subsequent assumption that internal energy is a state function, the total energy dissipated is determined from the integral

$$\Delta\gamma = \int_{t_i}^{t_f} \left[\boldsymbol{\tau}_f \bullet \mathbf{D} + \boldsymbol{\tau}_d \bullet \mathbf{D}_p + \boldsymbol{\Gamma}_c \bullet \mathbf{\Omega}_c + \boldsymbol{\Gamma}_\pi \bullet \boldsymbol{\pi} \right] dt \,.$$

The corresponding increase of internal energy is available from the special integral form (8.11) as

$$\Delta\varphi = \int_{t_i}^{t_f} \left\{ \left[\nabla_{D_p} \varphi\right]_r \bullet \mathbf{D}_p + \left[\nabla_{\Omega_e} \varphi\right]_r \bullet \mathbf{\Omega}_e + \left[\nabla_{\pi} \varphi\right]_r \bullet \boldsymbol{\pi} \right\} dt \,,$$

expressed in terms of the elastic recovery variables defined relative to the base state at $t = t_i$. In terms of the stress coefficient functions introduced in (7.4), it then follows that

$$\left[\nabla_{D_p} \varphi\right]_r = -\left[\boldsymbol{\tau}_d\right]_r \;;\quad \left[\nabla_{\Omega_e} \varphi\right]_r = -\left[\boldsymbol{\Gamma}_c\right]_r \;;\quad \left[\nabla_{\pi} \varphi\right]_r = -\left[\boldsymbol{\Gamma}_\pi\right]_r \,,$$

leading to the desired expression

$$\Delta\varphi = -\left[\int_{t_i}^{t_f} \left\{ \left[\boldsymbol{\tau}_d\right]_r \bullet \mathbf{D}_p + \left[\boldsymbol{\Gamma}_c\right]_r \bullet \mathbf{\Omega}_c + \left[\boldsymbol{\Gamma}_\pi\right]_r \bullet \boldsymbol{\pi} \right\} dt \right] ,$$

and the final integral form of the Il'iushin inequality (9.1) as

$$\int_{t_i}^{t_f} \boldsymbol{\tau} \bullet \mathbf{D}\, dt = \int_{t_i}^{t_f} \left[\boldsymbol{\tau}_f \bullet \mathbf{D} + \left\{ \boldsymbol{\tau}_d - [\boldsymbol{\tau}_d]_r \right\} \bullet \mathbf{D}_p + \left\{ \boldsymbol{\Gamma}_e - [\boldsymbol{\Gamma}_e]_r \right\} \bullet \boldsymbol{\Omega}_e + \left\{ \boldsymbol{\Gamma}_\pi - [\boldsymbol{\Gamma}_\pi]_r \right\} \bullet \boldsymbol{\pi} \right] dt \geq 0 .$$

Consideration shall now be restricted to *rate-independent*, *non-viscous*, *elastically compliant* solids. Recalling (7.2), (7.3)and (7.12), such materials are characterized by vanishing viscous stress

$$\boldsymbol{\tau}_f = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{\tau} = \boldsymbol{\tau}_e = \boldsymbol{\nabla}_D \varphi , \tag{9.2}$$

and inelastic mechanisms which are constrained by explicit *yield* criteria. As discussed in Section 7, this is expressed in terms of a state-dependent, scalar-valued *yield* function

$$\eta = \hat{\eta}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\eta}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \widehat{\eta}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}) \leq 0 ,$$

which defines a purely elastic state subspace $(\eta < 0)$, and restricts the activity of inelastic mechanisms through the elastic loading conditions (7.11). This function is, of course, necessarily subject to the same invariance criteria

$$\eta = \begin{cases} \hat{\eta}(\mathbf{Q}\mathbf{F}_e, \mathbf{T}_Q\boldsymbol{p}) = \tilde{\eta}(\mathbf{Q}\mathbf{v}\mathbf{Q}^T, \mathbf{Q}\mathbf{R}_e, \mathbf{T}_Q\boldsymbol{p}) = \widehat{\eta}(\mathbf{Q}\mathbf{a}\mathbf{Q}^T, \mathbf{Q}\mathbf{R}_e, \mathbf{T}_Q\boldsymbol{p}) \; ; \quad \text{for each } \mathbf{Q} \in \Theta , \\ \hat{\eta}(\mathbf{F}_e\mathbf{Q}^T, \boldsymbol{p}) = \tilde{\eta}(\mathbf{v}, \mathbf{R}_e\mathbf{Q}^T, \boldsymbol{p}) = \widehat{\eta}(\mathbf{a}, \mathbf{R}_e\mathbf{Q}^T, \boldsymbol{p}) \; ; \quad \text{for each } \mathbf{Q} \in \wp , \end{cases}$$

as all other constitutive functions. For such materials, the above Il'iushin inequality can be reexpressed in the convenient form

$$\varepsilon_{loss} = \Delta\varepsilon = \int_{t_i}^{t_f} \dot{\varepsilon}\, dt \;\geq\; 0 , \tag{9.3}$$

in terms of a dissipative-type energy rate

$$\dot{\varepsilon} \equiv \left\{ \boldsymbol{\tau}_d - [\boldsymbol{\tau}_d]_r \right\} \bullet \mathbf{D}_p + \left\{ \boldsymbol{\Gamma}_e - [\boldsymbol{\Gamma}_e]_r \right\} \bullet \boldsymbol{\Omega}_e + \left\{ \boldsymbol{\Gamma}_\pi - [\boldsymbol{\Gamma}_\pi]_r \right\} \bullet \boldsymbol{\pi} , \tag{9.4}$$

which clearly vanishes during any purely elastic loading segment.

Necessary conditions for Il'iushin stability can now be obtained by considering special closed deformation cycles. Before attempting this, a few preliminary observations and developments are required. In what follows, the collective variable $\boldsymbol{\mathcal{S}}$ shall be used to represent the time-varying material state

$$\boldsymbol{\mathcal{S}} = \begin{cases} \hat{\boldsymbol{\mathcal{S}}} = (\mathbf{F}_e, \boldsymbol{p}) \; ; \quad [\mathbf{F}_e = \mathbf{v}\mathbf{R}_e] , \\ \tilde{\boldsymbol{\mathcal{S}}} = (\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) , \\ \widehat{\boldsymbol{\mathcal{S}}} = (\mathbf{a}, \mathbf{R}_e, \boldsymbol{p}) , \end{cases}$$

and

$$\boldsymbol{\mathcal{S}}_r = \begin{cases} \hat{\boldsymbol{\mathcal{S}}}_r = (\mathbf{F}_{e_r}, \boldsymbol{p}) \; ; \quad [\mathbf{F}_{e_r} = \mathbf{v}_r\mathbf{R}_e] , \\ \tilde{\boldsymbol{\mathcal{S}}}_r = (\mathbf{v}_r, \mathbf{R}_e, \boldsymbol{p}) , \\ \widehat{\boldsymbol{\mathcal{S}}}_r = (\mathbf{a}_r, \mathbf{R}_e, \boldsymbol{p}) , \end{cases} \quad ; \quad [\mathbf{R}_{e_r} = \mathbf{R}_e \;\; \& \;\; \boldsymbol{p}_r = \boldsymbol{p}] ,$$

its corresponding *elastic recovery state* defined relative to the $t = t_i$ base state

$$\boldsymbol{\mathcal{S}}_r(t_i) = \boldsymbol{\mathcal{S}}(t_i) = \boldsymbol{\mathcal{S}}_o = \begin{cases} \hat{\boldsymbol{\mathcal{S}}}_o = (\mathbf{F}_{e_o}, \boldsymbol{p}_o) \; ; \quad [\mathbf{F}_{e_o} = \mathbf{v}_o\mathbf{R}_{e_o}] , \\ \tilde{\boldsymbol{\mathcal{S}}}_o = (\mathbf{v}_o, \mathbf{R}_{e_o}, \boldsymbol{p}_o) , \\ \widehat{\boldsymbol{\mathcal{S}}}_o = (\mathbf{a}_o, \mathbf{R}_{e_o}, \boldsymbol{p}_o) . \end{cases}$$

After recalling that all elastic recovery variables are observed to remain constant by shadow frame observers during purely elastic loading segments[20], it follows that

$$\mathcal{S}_r(t) \overset{\text{S.F.}}{=} \mathcal{S}_o \;;\quad t_i \le t \le t^* \tag{9.5}$$

during any initial elastic segment leading up to the inception of inelastic loading at $t=t^* \ge t_i$. Trivially, this observation extends to any state function

$$\Psi = \Psi(\mathcal{S}) = \hat{\Psi}(\mathbf{F}_e, \boldsymbol{p}) = \tilde{\Psi}(\mathbf{v}, \mathbf{R}_e, \boldsymbol{p}) = \widehat{\Psi}(\mathbf{a}, \mathbf{R}_e, \boldsymbol{p})$$

evaluated at the elastically recovered state

$$\Psi_r = \Psi(\mathcal{S}_r) = \hat{\Psi}(\mathbf{F}_{e_r}, \boldsymbol{p}) = \tilde{\Psi}(\mathbf{v}_r, \mathbf{R}_e, \boldsymbol{p}) = \widehat{\Psi}(\mathbf{a}_r, \mathbf{R}_e, \boldsymbol{p}),$$

as expressed by the relation

$$\Psi_r(t) \overset{\text{S.F.}}{=} \Psi_o = \Psi(\mathcal{S}_o) \;;\quad t_i \le t \le t^*. \tag{9.6}$$

Thus, the above dissipative energy rate (9.4) evaluates to

$$\dot{\varepsilon}\big|_{t=t^*} \overset{\text{S.F.}}{=} \left[\boldsymbol{\tau}_d(t^*) - \boldsymbol{\tau}_{d_o}\right] \bullet \mathbf{D}_p(t^*) + \left[\boldsymbol{\Gamma}_e(t^*) - \boldsymbol{\Gamma}_{e_o}\right] \bullet \boldsymbol{\Omega}_e(t^*) + \left[\boldsymbol{\Gamma}_\pi(t^*) - \boldsymbol{\Gamma}_{\pi_o}\right] \bullet \boldsymbol{\pi}(t^*) \tag{9.7}$$

at the inception of inelastic loading at $t=t^* \ge t_i$.

It is also necessary to express the assumed yield criterion (7.11) in a more explicit form. To facilitate this, let

$$\mathcal{E}_o = \mathcal{E}(\mathbf{R}_{e_o}, \boldsymbol{p}_o) = \{\mathcal{S} : \mathbf{R}_e = \mathbf{R}_{e_o}, \boldsymbol{p} = \boldsymbol{p}_o \ \langle\text{and}\rangle\ \eta(\mathcal{S}) < 0\},$$
$$\partial\mathcal{E}_o = \partial\mathcal{E}(\mathbf{R}_{e_o}, \boldsymbol{p}_o) = \{\mathcal{S} : \mathbf{R}_e = \mathbf{R}_{e_o}, \boldsymbol{p} = \boldsymbol{p}_o \ \langle\text{and}\rangle\ \eta(\mathcal{S}) = 0\},$$

denote the *elastic* and *elastic-plastic transition* regions corresponding to the dislocation components $\boldsymbol{p} = \boldsymbol{p}_o$, and reference cell orientation $\mathbf{R}_e = \mathbf{R}_{e_o}$. Also, recall that the classical yield formulation is based on the assumption that the **inelastic mechanisms act, only as needed, to maintain the yield inequality $\eta \le 0$, and thereby to insure the possibility of elastic "unloading" from any accessible state, and of sustained plastic loading through any elastic-plastic transition state**. In other words, it is generally assumed that any prescribed deformation that is "elastically compatible" with $\eta \le 0$, will proceed elastically, and that any process which is not, will maintain the transition state condition $\eta = 0$. In terms of the *elastic* and *non-elastic* shadow rates (6.2), this standard yield criterion takes the form

$$\begin{array}{lcl} \left\{ \begin{array}{c} \mathcal{S} \in \mathcal{E}_o \\ \langle\text{or}\rangle \\ \mathcal{S} \in \partial\mathcal{E}_o \ \& \ \overset{e}{\dot{\eta}} \le 0 \end{array} \right\} & \Rightarrow & \{(\mathbf{D}_p, \boldsymbol{\Omega}_e, \boldsymbol{\pi}) = \mathbf{0}\} \\ \left\{ \mathcal{S} \in \partial\mathcal{E}_o \ \& \ \overset{e}{\dot{\eta}} > 0 \right\} & \Rightarrow & \left\{ \begin{array}{c} (\mathbf{D}_p, \boldsymbol{\Omega}_e, \boldsymbol{\pi}) \ne \mathbf{0} \\ \langle\text{and}\rangle \\ \overset{s}{\dot{\eta}} = 0 \end{array} \right\} \end{array} \tag{9.8}$$

[20] Refer to property (**iv**) immediately following (8.9).

expressed in terms of the elastic component $(6.7)_1$ of the shadow rate derivative of the yield function.

Now, consider a closed deformation path $(t_i \leq t \leq t_f)$ for which eventual elastic recovery (at $t = t_f$) is preceded by an elastic segment $(t_i \leq t \leq t_1)$ leading from an initial state $\mathcal{S}_o \in \mathcal{E}_o$ to a transition state $\mathcal{S}_1$ with $\eta(\mathcal{S}_1) = 0$, followed by an infinitesimal segment $[t_1 \leq t \leq (t_1 + \Delta t)]$ of inelastic loading $(\overset{e}{\eta} > 0)$. Observed from the shadow frame, the result (9.5) guarantees that

$$\mathcal{S}_r(t_1) \overset{\text{S.F.}}{=} \mathcal{S}_o \quad \Rightarrow \quad \begin{cases} \mathbf{R}_{c_r}(t_1) = \mathbf{R}_e(t_1) = \mathbf{R}_e(t_i) = \mathbf{R}_{c_o}, \\ \boldsymbol{p}_r(t_1) = \boldsymbol{p}(t_1) = \boldsymbol{p}(t_i) = \boldsymbol{p}_o, \end{cases} \tag{9.9}$$

which, in turn, provides that

$$[\mathcal{S}_1]_{\text{S.F.}} \in \partial\mathcal{E}_o = \{\mathcal{S} : \mathbf{R}_e = \mathbf{R}_{c_o}, \boldsymbol{p} = \boldsymbol{p}_o \ \langle\text{and}\rangle \ \eta(\mathcal{S}) = 0\}.$$

Moreover, for such a cycle, the Il'iushin inequality (9.3) takes the form

$$\varepsilon_{\text{loss}} = \int_{t_1}^{t_1 + \Delta t} \dot{\varepsilon}\, dt = \dot{\varepsilon}_1 \Delta t + \cdots \geq 0 \ ; \quad \dot{\varepsilon}_1 \equiv \dot{\varepsilon}\big|_{t = t_1}$$

when expanded in powers of the small (infinitesimal) inelastic time increment Δt. With this, and the special shadow frame result (9.7), it is evident that the inequality

$$\boxed{(\boldsymbol{\tau}_{d_1} - \boldsymbol{\tau}_{d_o}) \bullet \mathbf{D}_{p_1} + (\boldsymbol{\Gamma}_{c_1} - \boldsymbol{\Gamma}_{c_o}) \bullet \boldsymbol{\Omega}_{e_1} + (\boldsymbol{\Gamma}_{\pi_1} - \boldsymbol{\Gamma}_{\pi_o}) \bullet \boldsymbol{\pi}_1 \geq 0}, \tag{9.10}$$

for all $\mathcal{S}_o \in \mathcal{E}_o$, and all possible inelastic rates $(\mathbf{D}_{p_1}, \boldsymbol{\Omega}_{e_1}, \boldsymbol{\pi}_1)$ leading from $\mathcal{S}_1 \in \partial\mathcal{E}_o$, is a *necessary condition* for Il'iushin stability.

A second requirement is obtained by considering an infinitesimal deformation cycle consisting of a single inelastic loading segment of duration Δt leading from an initial transition state $\mathcal{S}_o \in \partial\mathcal{E}_o$, followed by elastic recovery. Direct application of the results (9.6) and (8.10) confirms the shadow frame perceptions

$$\begin{aligned} \Psi_r(t_i) &\overset{\text{S.F.}}{=} \Psi_o = \Psi(t_i) \quad \Rightarrow \quad [\Psi - \Psi_r]_{t = t_i} \overset{\text{S.F.}}{=} 0, \\ \overset{s}{\Psi}_r(t_i) &\overset{\text{S.F.}}{=} \overset{n}{\Psi}(t_i) = \overset{s}{\Psi}(t_i) - \overset{e}{\Psi}(t_i) \quad \Rightarrow \quad \overline{(\Psi - \Psi_r)}^{\,s}_{t = t_i} \overset{\text{S.F.}}{=} \overset{e}{\Psi}(t_i), \end{aligned} \tag{9.11}$$

at the inception $(t = t_i)$ of such a process. Observe that these identities, together with the inelastic loading criteria from (9.8), also insure the existence of such cycles inasmuch as the inequality

$$\overset{s}{\eta}_r = \overset{n}{\eta} = \overset{s}{\eta} - \overset{e}{\eta} = -\overset{e}{\eta} < 0$$

guarantees that the elastic recovery (terminal) state lies within the updated elastic region. Moreover, as a consequence of $(9.11)_1$, it is clear that $\dot{\varepsilon}_i = 0$ and hence the energy loss inequality (9.3) expands to

$$\varepsilon_{\text{loss}} = \int_{t_i}^{t_i + \Delta t} \dot{\varepsilon}\, dt = \tfrac{1}{2} \ddot{\varepsilon}_i (\Delta t)^2 + \cdots \geq 0 \ ; \quad \ddot{\varepsilon}_i \equiv \ddot{\varepsilon}\big|_{t = t_i}.$$

Shadow frame differentiation of (9.4), in view of the above relations (9.11), then leads to a second necessary condition for Il'iushin stability, namely that

$$\boxed{\overset{e}{\boldsymbol{\tau}}_d \bullet \mathbf{D}_p + \overset{e}{\boldsymbol{\Gamma}}_c \bullet \boldsymbol{\Omega}_c + \overset{e}{\boldsymbol{\Gamma}}_{\pi} \bullet \boldsymbol{\pi} \geq 0} \tag{9.12}$$

during inelastic loading.

Recall that the above derived stability conditions (9.10) and (9.12) apply to *rate independent* (and therefore *non-viscous*), *elastically compliant* solids. Further specialization results in further simplification. For example, if the material also has *invariant elastic properties,* then the results (7.13), $(9.9)_2$, $(6.3)_1$, $(6.6)_4$ and $(7.4)_3$ guarantee that the stress and stress rate coefficients of the dislocation rate $\boldsymbol{\pi}$ in both inequalities vanish. If the material is *structurally isotropic*, then, in view of (7.15), (7.17) and (9.2), all terms containing the 'cell rotation stress' $\boldsymbol{\Gamma}_c$ and cell spin rate $\boldsymbol{\Omega}_c$ vanish, and the *dissipative*, *elastic* and *true* stresses are identical, *i.e.* $\boldsymbol{\tau}=\boldsymbol{\tau}_d=\boldsymbol{\tau}_c$. These observations are significant in light of the seminal work of Drucker (1951), since the above Il'iushin inequalities are seen to reduce to the Drucker-like forms

$$\boxed{\begin{gathered}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_o)\bullet\mathbf{D}_{p_1}\geq 0\\ \overset{e}{\boldsymbol{\tau}}\bullet\mathbf{D}_p\geq 0\end{gathered}} \tag{9.13}$$

for all *rate independent* (*non-viscous*), *elastically compliant*, *structurally isotropic* materials which also have *invariant elastic properties* - no matter how complex their dislocation state space[21]. It is important to note that these are expressed in terms of the Kirchhoff (rather than Cauchy) stress and are rigorously confirmed at arbitrary levels of elastic and/or plastic strain. This result serves to verify the stability inequalities derived in Dashner (1986c) for large deformation, isotropic hardening elasto-plasticity.

The development of fully Eulerian CFD-type flow equations modeling various large deformation forming processes for this special class of materials is but one potential application of these theoretical results.

Appendix. Some fourth order friends

Throughout this appendix, the symbols $\mathcal{T}$ and $\mathcal{S}$ shall be used to represent the (vector) space of all second order tensors, and its linear subspace of all *symmetric* second order tensors respectively, that is

$$\mathcal{T}\sim \text{9-dim'l linear space of 2nd order (Euclidean) tensors},$$
$$\mathcal{S}\sim \text{6-dim'l linear space of } symmetric \text{ 2nd order tensors}.$$

Now, for any given $\mathbf{A}\in\mathcal{S}$, the expressions

$$\begin{aligned}&\underset{\sim}{\mathbf{S}}_A:\mathcal{T}\to\mathcal{T};\\ &\quad\underset{\sim}{\mathbf{S}}_A\mathbf{X}\equiv\tfrac{1}{2}(\mathbf{AX}+\mathbf{XA})\in\mathcal{T};\quad\text{for each }\mathbf{X}\in\mathcal{T},\\ &\underset{\sim}{\mathbf{A}}_A:\mathcal{T}\to\mathcal{T};\\ &\quad\underset{\sim}{\mathbf{A}}_A\mathbf{X}\equiv\tfrac{1}{2}(\mathbf{AX}-\mathbf{XA})\in\mathcal{T};\quad\text{for each }\mathbf{X}\in\mathcal{T},\\ &\underset{\sim}{\mathbf{B}}_A:\mathcal{T}\to\mathcal{T};\\ &\quad\underset{\sim}{\mathbf{B}}_A\mathbf{X}\equiv\mathbf{AXA}\in\mathcal{T};\quad\text{for each }\mathbf{X}\in\mathcal{T}.\end{aligned} \tag{A.1}$$[22]

define linear transformations on the (vector) space of tensors $\mathcal{T}$. The above defined mappings shall be

[21] For materials of this type, any number of state variables could be introduced for the purpose of modeling the size and shape of the yield surface, and for incorporating dislocation induced effects into the plastic flow relation. Classical elasto-plastic models which incorporate isotropic and kinematic hardening are of this type.

referred to (respectively) as the "symmetric product" (with **A**), the "antisymmetric product" (with **A**), and the "**A**-bracket" functions.

The *symmetric product*, *antisymmetric product*, and *bracket* functions defined above (for $\mathbf{A}\in\mathcal{S}$) are *double tensors* (4th-order tensors), that is, linear homogeneous (degree one) maps from the space of second order tensors into itself. In view of the well-known identities[23]

$$\left.\begin{aligned}(\mathbf{XY})^{\mathrm{T}} &= \mathbf{Y}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\\ \mathbf{X}\bullet\mathbf{Y} &= tr\left(\mathbf{X}^{\mathrm{T}}\mathbf{Y}\right)\\ \mathbf{X}\bullet\mathbf{Y} &= \mathbf{X}^{\mathrm{T}}\bullet\mathbf{Y}^{\mathrm{T}} = \mathbf{Y}\bullet\mathbf{X} = \mathbf{Y}^{\mathrm{T}}\bullet\mathbf{X}^{\mathrm{T}}\\ \mathbf{X}\bullet(\mathbf{YZ}) &= \mathbf{Y}\bullet\left(\mathbf{XZ}^{\mathrm{T}}\right) = \mathbf{Z}\bullet\left(\mathbf{Y}^{\mathrm{T}}\mathbf{X}\right)\end{aligned}\right\} ;\quad (\mathbf{X},\mathbf{Y},\mathbf{Z})\subset\mathcal{T},$$

it is a simple exercise to show that these operators conform to the algebraic identities

$$\underset{\sim}{\mathbf{T}} = \underset{\sim}{\mathbf{T}}^{\mathrm{T}} \;\Leftrightarrow\; \mathbf{X}\bullet\left(\underset{\sim}{\mathbf{T}}\mathbf{Y}\right) = \mathbf{Y}\bullet\left(\underset{\sim}{\mathbf{T}}\mathbf{X}\right) \;;\; \underset{\sim}{\mathbf{T}}\in\left[\underset{\sim}{\mathbf{S}}_{A},\underset{\sim}{\mathbf{A}}_{A},\underset{\sim}{\mathbf{B}}_{A}\right], \tag{A.2}$$

$$\begin{gathered}\left(\underset{\sim}{\mathbf{S}}_{A}\mathbf{X}\right)^{\mathrm{T}} = \underset{\sim}{\mathbf{S}}_{A}\mathbf{X}^{\mathrm{T}} \quad \& \quad \left(\underset{\sim}{\mathbf{B}}_{A}\mathbf{X}\right)^{\mathrm{T}} = \underset{\sim}{\mathbf{B}}_{A}\mathbf{X}^{\mathrm{T}},\\ \left(\underset{\sim}{\mathbf{A}}_{A}\mathbf{X}\right)^{\mathrm{T}} = -\left(\underset{\sim}{\mathbf{A}}_{A}\mathbf{X}^{\mathrm{T}}\right),\end{gathered} \tag{A.3}$$

and

$$\begin{gathered}\underset{\sim}{\mathbf{S}}_{A}\circ\underset{\sim}{\mathbf{A}}_{A} = \underset{\sim}{\mathbf{A}}_{A}\circ\underset{\sim}{\mathbf{S}}_{A} = \tfrac{1}{2}\underset{\sim}{\mathbf{A}}_{A^2},\\ \left.\begin{aligned}\underset{\sim}{\mathbf{S}}_{A}\mathbf{B} &= \underset{\sim}{\mathbf{S}}_{B}\mathbf{A}\\ \underset{\sim}{\mathbf{A}}_{A}\mathbf{B} &= -\underset{\sim}{\mathbf{A}}_{B}\mathbf{A}\end{aligned}\right\} ;\quad \mathbf{B}\in\mathcal{S},\\ \left.\begin{aligned}\underset{\sim}{\mathbf{B}}_{A}^{-1} &= \underset{\sim}{\mathbf{B}}_{A^{-1}}\\ \underset{\sim}{\mathbf{B}}_{A}\circ\underset{\sim}{\mathbf{S}}_{A^{-1}} &= \underset{\sim}{\mathbf{S}}_{A^{-1}}\circ\underset{\sim}{\mathbf{B}}_{A} = \underset{\sim}{\mathbf{S}}_{A}\\ \underset{\sim}{\mathbf{B}}_{A}\circ\underset{\sim}{\mathbf{A}}_{A^{-1}} &= \underset{\sim}{\mathbf{A}}_{A^{-1}}\circ\underset{\sim}{\mathbf{B}}_{A} = -\underset{\sim}{\mathbf{A}}_{A}\end{aligned}\right\} ;\quad \text{nonsingular}\,\mathbf{A}\in\mathcal{S},\end{gathered} \tag{A.4}$$

for any $\mathbf{A}\in\mathcal{S}$ and all tensor pairs $(\mathbf{X},\mathbf{Y})\subset\mathcal{T}$. The first set of relations establishes that each of these operators is a *symmetric* bilinear form on $\mathcal{T}$. The second set guarantees that the operators $\underset{\sim}{\mathbf{S}}_{A}$ and $\underset{\sim}{\mathbf{B}}_{A}$ preserve the (anti)symmetry of their respective arguments in that they map the symmetric and antisymmetric tensor subspaces into themselves. Conversely, the operator $\underset{\sim}{\mathbf{A}}_{A}$ maps symmetric/antisymmetric tensors into antisymmetric/symmetric tensors. The third set of relations document a number of easily established algebraic/composition relations involving the three operators.

A more comprehensive statement of operator commutivity is also available. Beginning with the selection of any symmetric pair $(\mathbf{A},\mathbf{B})\subset\mathcal{S}$ which commute, i.e.

$$\mathbf{AB} = \mathbf{BA} \;\Leftrightarrow\; \underset{\sim}{\mathbf{A}}_{A}\mathbf{B} = -\underset{\sim}{\mathbf{A}}_{B}\mathbf{A} = \mathbf{0},$$

it is easily established that

$$\underset{\sim}{\mathbf{T}}_{1}\circ\underset{\sim}{\mathbf{T}}_{2} = \underset{\sim}{\mathbf{T}}_{2}\circ\underset{\sim}{\mathbf{T}}_{1} \tag{A.5}$$

for <u>any</u> pair of operators

---

[22] A comprehensive examination of the operators $\underset{\sim}{\mathbf{S}}_{A}$ and $\underset{\sim}{\mathbf{A}}_{A}$ for <u>any</u> $\mathbf{A}\in\mathcal{T}$ is contained in the work of Scheidler (1994) and Guo *et al.* (1992). The present abbreviated development, while specialized for the case where $\mathbf{A}\in\mathcal{S}$, is adequate for the purposes of this paper.

[23] These shall henceforth be assumed known and freely used without citation.

$$\left(\underset{\sim}{\mathbf{T}}_1, \underset{\sim}{\mathbf{T}}_2\right) \subset \left[\underset{\sim}{\mathbf{S}}_A, \underset{\sim}{\mathbf{A}}_A, \underset{\sim}{\mathbf{B}}_A, \underset{\sim}{\mathbf{S}}_B, \underset{\sim}{\mathbf{A}}_B, \underset{\sim}{\mathbf{B}}_B\right].$$

While the above conclusions follow from simple algebraic manipulations of the defining expressions (A.1), further information is revealed through examination of the matrix representations for these operators relative to a special tensor basis. For this, let $\{\mathrm{A}_k, \hat{\mathrm{A}}_k\}_{k=1}^{3}$ represent the triad of scalar/vector eigenpairs for some symmetric tensor $\mathbf{A} \in \mathcal{S}$, and note that the set of nine (9) tensors $\{\mathbf{A}_\alpha\}_{\alpha=1}^{9} \subset \mathcal{T}$

$$\begin{aligned}
\mathbf{A}_k &\equiv \hat{\mathrm{A}}_k \otimes \hat{\mathrm{A}}_k \; ; \quad k = 1,2,3 , \\
\mathbf{A}_4 &\equiv \hat{\mathrm{A}}_2 \otimes \hat{\mathrm{A}}_3 \; ; \quad \mathbf{A}_5 \equiv \hat{\mathrm{A}}_3 \otimes \hat{\mathrm{A}}_2 , \\
\mathbf{A}_6 &\equiv \hat{\mathrm{A}}_3 \otimes \hat{\mathrm{A}}_1 \; ; \quad \mathbf{A}_7 \equiv \hat{\mathrm{A}}_1 \otimes \hat{\mathrm{A}}_3 , \\
\mathbf{A}_8 &\equiv \hat{\mathrm{A}}_1 \otimes \hat{\mathrm{A}}_2 \; ; \quad \mathbf{A}_9 \equiv \hat{\mathrm{A}}_2 \otimes \hat{\mathrm{A}}_1 ,
\end{aligned} \tag{A.6}$$

forms an orthonormal basis,

$$\mathbf{A}_\alpha \bullet \mathbf{A}_\beta = \delta_{\alpha\beta} = \begin{cases} 1 \; ; & \text{whenever } \alpha = \beta , \\ 0 \; ; & \text{whenever } \alpha \neq \beta , \end{cases} \tag{A.7}$$

for the nine (9) dimensional 'vector space' of second order tensors $\mathcal{T}$, with

$$\begin{aligned}
\mathbf{X} &= \sum_{i=1}^{3} \sum_{j=1}^{3} \mathrm{X}_{ij} \left(\hat{\mathrm{A}}_i \otimes \hat{\mathrm{A}}_j\right) \\
\mathbf{X} &= \mathrm{X}_{11}\mathbf{A}_1 + \mathrm{X}_{22}\mathbf{A}_2 + \mathrm{X}_{33}\mathbf{A}_3 + \mathrm{X}_{23}\mathbf{A}_4 + \mathrm{X}_{32}\mathbf{A}_5 + \mathrm{X}_{31}\mathbf{A}_6 + \mathrm{X}_{13}\mathbf{A}_7 + \mathrm{X}_{12}\mathbf{A}_8 + \mathrm{X}_{21}\mathbf{A}_9 , \\
\Rightarrow \quad [\![\mathbf{X}]\!]_{\mathbf{A}_\alpha} &= \mathit{column}\left(\mathrm{X}_{11}, \mathrm{X}_{22}, \mathrm{X}_{33}, \mathrm{X}_{23}, \mathrm{X}_{32}, \mathrm{X}_{31}, \mathrm{X}_{13}, \mathrm{X}_{12}, \mathrm{X}_{21}\right) ,
\end{aligned} \tag{A.8}$$

for any tensor element $\mathbf{X} \in \mathcal{T}$. Relative to this special $\mathcal{T}$-basis, any fourth order tensor $\underset{\sim}{\mathbf{T}} : \mathcal{T} \rightarrow \mathcal{T}$ admits a unique dyadic representation of the form

$$\underset{\sim}{\mathbf{T}} = \sum_{\alpha=1}^{9} \sum_{\beta=1}^{9} \mathrm{T}_{\alpha\beta} \left(\mathbf{A}_\alpha \otimes \mathbf{A}_\beta\right) , \tag{A.9}$$

expressed in terms of the scalar coefficients

$$\mathrm{T}_{\alpha\beta} = \mathbf{A}_\alpha \bullet \left(\underset{\sim}{\mathbf{T}} \cdot \mathbf{A}_\beta\right) \; ; \quad \alpha, \beta = 1,2,\ldots,9 .$$

With reference to (A.1), (A.6) - (A.9), and the standard eigenbasis expansions

$$\left[\underset{\sim}{\mathbf{S}}_A \cdot \mathbf{X}\right]_{\hat{\mathrm{A}}_k} = \frac{1}{2} \left[\begin{pmatrix} \mathrm{A}_1 & 0 & 0 \\ 0 & \mathrm{A}_2 & 0 \\ 0 & 0 & \mathrm{A}_3 \end{pmatrix} \begin{pmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} & \mathrm{X}_{13} \\ \mathrm{X}_{21} & \mathrm{X}_{22} & \mathrm{X}_{23} \\ \mathrm{X}_{31} & \mathrm{X}_{32} & \mathrm{X}_{33} \end{pmatrix} + \begin{pmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} & \mathrm{X}_{13} \\ \mathrm{X}_{21} & \mathrm{X}_{22} & \mathrm{X}_{23} \\ \mathrm{X}_{31} & \mathrm{X}_{32} & \mathrm{X}_{33} \end{pmatrix} \begin{pmatrix} \mathrm{A}_1 & 0 & 0 \\ 0 & \mathrm{A}_2 & 0 \\ 0 & 0 & \mathrm{A}_3 \end{pmatrix} \right]$$

$$\left[\underset{\sim}{\mathbf{S}}_A \cdot \mathbf{X}\right]_{\hat{\mathrm{A}}_k} = \begin{pmatrix} \mathrm{A}_1\mathrm{X}_{11} & \frac{1}{2}(\mathrm{A}_1+\mathrm{A}_2)\mathrm{X}_{12} & \frac{1}{2}(\mathrm{A}_3+\mathrm{A}_1)\mathrm{X}_{13} \\ \frac{1}{2}(\mathrm{A}_1+\mathrm{A}_2)\mathrm{X}_{21} & \mathrm{A}_2\mathrm{X}_{22} & \frac{1}{2}(\mathrm{A}_2+\mathrm{A}_3)\mathrm{X}_{23} \\ \frac{1}{2}(\mathrm{A}_3+\mathrm{A}_1)\mathrm{X}_{31} & \frac{1}{2}(\mathrm{A}_2+\mathrm{A}_3)\mathrm{X}_{32} & \mathrm{A}_3\mathrm{X}_{33} \end{pmatrix} , \tag{A.10}$$

....

$$\left[\underset{\sim}{\mathbf{A}}_A \cdot \mathbf{X}\right]_{\hat{\mathrm{A}}_k} = \frac{1}{2} \left[\begin{pmatrix} \mathrm{A}_1 & 0 & 0 \\ 0 & \mathrm{A}_2 & 0 \\ 0 & 0 & \mathrm{A}_3 \end{pmatrix} \begin{pmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} & \mathrm{X}_{13} \\ \mathrm{X}_{21} & \mathrm{X}_{22} & \mathrm{X}_{23} \\ \mathrm{X}_{31} & \mathrm{X}_{32} & \mathrm{X}_{33} \end{pmatrix} - \begin{pmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} & \mathrm{X}_{13} \\ \mathrm{X}_{21} & \mathrm{X}_{22} & \mathrm{X}_{23} \\ \mathrm{X}_{31} & \mathrm{X}_{32} & \mathrm{X}_{33} \end{pmatrix} \begin{pmatrix} \mathrm{A}_1 & 0 & 0 \\ 0 & \mathrm{A}_2 & 0 \\ 0 & 0 & \mathrm{A}_3 \end{pmatrix} \right]$$

$$[\underset{\sim}{\mathbf{A}}_A \mathbf{X}]_{\hat{\mathbf{A}}_k} = \begin{pmatrix} 0 & \frac{1}{2}(A_1-A_2)X_{12} & -\frac{1}{2}(A_3-A_1)X_{13} \\ -\frac{1}{2}(A_1-A_2)X_{21} & 0 & \frac{1}{2}(A_2-A_3)X_{23} \\ \frac{1}{2}(A_3-A_1)X_{31} & -\frac{1}{2}(A_2-A_3)X_{32} & 0 \end{pmatrix}, \quad (A.11)$$

$$[\underset{\sim}{\mathbf{B}}_A \mathbf{X}]_{\hat{\mathbf{A}}_k} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$$

$$[\underset{\sim}{\mathbf{B}}_A \mathbf{X}]_{\hat{\mathbf{A}}_k} = \begin{pmatrix} A_1^2X_{11} & (A_1A_2)X_{12} & (A_3A_1)X_{13} \\ (A_1A_2)X_{21} & A_2^2X_{22} & (A_2A_3)X_{23} \\ (A_3A_1)X_{31} & (A_2A_3)X_{32} & A_3^2X_{33} \end{pmatrix}, \quad (A.12)$$

it is a simple matter to establish the diagonal matrix representations

$$\{S_{\alpha\beta}\}_{9\times 9} = [\![\underset{\sim}{\mathbf{S}}_A]\!]_{\mathbf{A}_\alpha} = diagonal[A_1, A_2, A_3, \tfrac{1}{2}(A_2+A_3), \tfrac{1}{2}(A_2+A_3), \tfrac{1}{2}(A_3+A_1), \tfrac{1}{2}(A_3+A_1), \tfrac{1}{2}(A_1+A_2), \tfrac{1}{2}(A_1+A_2)]$$
$$= diagonal\left[A_1, A_2, A_3, \tfrac{1}{2}(A_2+A_3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tfrac{1}{2}(A_3+A_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tfrac{1}{2}(A_1+A_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right],$$

$$\{A_{\alpha\beta}\}_{9\times 9} = [\![\underset{\sim}{\mathbf{A}}_A]\!]_{\mathbf{A}_\alpha} = diagonal[0, 0, 0, \tfrac{1}{2}(A_2-A_3), -\tfrac{1}{2}(A_2-A_3), \tfrac{1}{2}(A_3-A_1), -\tfrac{1}{2}(A_3-A_1), \tfrac{1}{2}(A_1-A_2), -\tfrac{1}{2}(A_1-A_2)]$$
$$= diagonal\left[0, 0, 0, \tfrac{1}{2}(A_2-A_3)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tfrac{1}{2}(A_3-A_1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tfrac{1}{2}(A_1-A_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right], \quad (A.13)$$

$$\{B_{\alpha\beta}\}_{9\times 9} = [\![\underset{\sim}{\mathbf{B}}_A]\!]_{\mathbf{A}_\alpha} = diagonal[A_1^2, A_2^2, A_3^2, A_2A_3, A_2A_3, A_3A_1, A_3A_1, A_1A_2, A_1A_2]$$
$$= diagonal\left[A_1^2, A_2^2, A_3^2, A_2A_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_3A_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1A_2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

In addition to providing confirmation for the various properties listed above, these (diagonal) matrix forms can also be used to establish additional properties. For example, the operator $\underset{\sim}{\mathbf{S}}_A$ is clearly seen to be *non-singular* and therefore *invertible* with

$$\{S^{-1}_{\alpha\beta}\}_{9\times 9} = [\![\underset{\sim}{\mathbf{S}}^{-1}_A]\!]_{\mathbf{A}_\alpha} = diagonal\left[\frac{1}{A_1}, \frac{1}{A_2}, \frac{1}{A_3}, \frac{2}{A_2+A_3}, \frac{2}{A_2+A_3}, \frac{2}{A_3+A_1}, \frac{2}{A_3+A_1}, \frac{2}{A_1+A_2}, \frac{2}{A_1+A_2}\right]$$
$$= diagonal\left[\frac{1}{A_1}, \frac{1}{A_2}, \frac{1}{A_3}, \frac{2}{A_2+A_3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{2}{A_3+A_1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{2}{A_1+A_2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right],$$

whenever the **A**-eigenvalues satisfy

$$A_1A_2A_3(A_2+A_3)(A_3+A_1)(A_1+A_2) \neq 0.$$

It is noteworthy that this condition is guaranteed whenever $\mathbf{A} \in \mathcal{S}$ is *definite*, either positive or negative. For present purposes, it will suffice to state that, **for *positive definite* $\mathbf{A} \in \mathcal{S}$, the operator $\underset{\sim}{\mathbf{S}}_A$ and its *inverse* $\underset{\sim}{\mathbf{S}}^{-1}_A$ are both *positive definite*, *symmetric* bilinear forms on $\mathcal{T}$**

$$\mathbf{X}\bullet(\underset{\sim}{\mathbf{S}}^{\pm 1}_A \mathbf{X}) \left\{ \begin{matrix} = 0 \; ; & \mathbf{X} = \mathbf{0} \\ > 0 \; ; & \text{otherwise} \end{matrix} \right\rangle ; \quad \mathbf{X} \in \mathcal{T},$$
$$\underset{\sim}{\mathbf{S}}^{\pm 1}_A = \underset{\sim}{\mathbf{S}}^{\pm T}_A \Leftrightarrow \mathbf{X}\bullet(\underset{\sim}{\mathbf{S}}^{\pm 1}_A \mathbf{Y}) = \mathbf{Y}\bullet(\underset{\sim}{\mathbf{S}}^{\pm 1}_A \mathbf{X}) \; ; \quad (\mathbf{X}, \mathbf{Y}) \subset \mathcal{T}, \quad (A.14)$$

both map symmetric/antisymmetric tensors into symmetric/antisymmetric tensors

$$\left(\underset{\sim}{\mathbf{S}}_A^{\pm 1}\cdot\mathbf{X}\right)^T = \underset{\sim}{\mathbf{S}}_A^{\pm 1}\cdot\mathbf{X}^T \;;\quad \mathbf{X}\in\mathcal{T}, \tag{A.15}$$

and have the respective diagonal eigenbasis matrix representations

$$\begin{aligned}\left[\!\left[\underset{\sim}{\mathbf{S}}_A\right]\!\right]_{A_\alpha} &= diagonal\left[A_1, A_2, A_3, \frac{A_2+A_3}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{A_3+A_1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{A_1+A_2}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right],\\ \left[\!\left[\underset{\sim}{\mathbf{S}}_A^{-1}\right]\!\right]_{A_\alpha} &= diagonal\left[\frac{1}{A_1}, \frac{1}{A_2}, \frac{1}{A_3}, \frac{2}{A_2+A_3}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{2}{A_3+A_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{2}{A_1+A_2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right].\end{aligned} \tag{A.16}$$

The above **A**-bracket property $(A.4)_4$ can now be similarly updated to state that, **for any *definite* $\mathbf{A}\in\mathcal{S}$, the operator $\underset{\sim}{\mathbf{B}}_A$ and its inverse $\underset{\sim}{\mathbf{B}}_A^{-1} = \underset{\sim}{\mathbf{B}}_{A^{-1}}$ are both *positive definite, symmetric* bilinear forms on $\mathcal{T}$**

$$\begin{aligned}&\mathbf{X}\bullet\left(\underset{\sim}{\mathbf{B}}_A^{\pm 1}\cdot\mathbf{X}\right)\left\{\begin{array}{ll} =\mathbf{0} \;; & \mathbf{X}=\mathbf{0} \\ >\mathbf{0} \;; & \text{otherwise}\end{array}\right\} ;\quad \mathbf{X}\in\mathcal{T},\\ &\underset{\sim}{\mathbf{B}}_A^{\pm 1} = \underset{\sim}{\mathbf{B}}_A^{\pm T} \;\Leftrightarrow\; \mathbf{X}\bullet\left(\underset{\sim}{\mathbf{B}}_A^{\pm 1}\cdot\mathbf{Y}\right) = \mathbf{Y}\bullet\left(\underset{\sim}{\mathbf{B}}_A^{\pm 1}\cdot\mathbf{X}\right) \;;\quad (\mathbf{X},\mathbf{Y})\subset\mathcal{T},\end{aligned} \tag{A.17}$$

both map symmetric/antisymmetric tensors into symmetric/antisymmetric tensors

$$\left(\underset{\sim}{\mathbf{B}}_A^{\pm 1}\cdot\mathbf{X}\right)^T = \underset{\sim}{\mathbf{B}}_A^{\pm 1}\cdot\mathbf{X}^T \;;\quad \mathbf{X}\in\mathcal{T}, \tag{A.18}$$

and have the respective diagonal eigenbasis matrix representations

$$\begin{aligned}\left[\!\left[\underset{\sim}{\mathbf{B}}_A\right]\!\right]_{A_\alpha} &= diagonal\left[A_1^2, A_2^2, A_3^2, A_2A_3\begin{pmatrix}1&0\\0&1\end{pmatrix}, A_3A_1\begin{pmatrix}1&0\\0&1\end{pmatrix}, A_1A_2\begin{pmatrix}1&0\\0&1\end{pmatrix}\right],\\ \left[\!\left[\underset{\sim}{\mathbf{B}}_A^{-1}\right]\!\right]_{A_\alpha} &= diagonal\left[\frac{1}{A_1^2}, \frac{1}{A_2^2}, \frac{1}{A_3^2}, \frac{1}{A_2A_3}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{1}{A_3A_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{1}{A_1A_2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right].\end{aligned} \tag{A.19}$$

In addition, since any pair of symmetric tensors which commute share a common eigenbasis, the above listed commutative property (A.5) may also be updated to state that, **for any symmetric pair $(\mathbf{A},\mathbf{B})\subset\mathcal{S}$ which commute, that is**

$$\mathbf{AB}=\mathbf{BA} \;\Leftrightarrow\; \underset{\sim}{\mathbf{A}}_A\cdot\mathbf{B} = -\underset{\sim}{\mathbf{A}}_B\cdot\mathbf{A} = \mathbf{0},$$

it follows that

$$\underset{\sim}{\mathbf{T}}_1\circ\underset{\sim}{\mathbf{T}}_2 = \underset{\sim}{\mathbf{T}}_2\circ\underset{\sim}{\mathbf{T}}_1 \tag{A.20}$$

for any (existing) operator pair

$$\left(\underset{\sim}{\mathbf{T}}_1, \underset{\sim}{\mathbf{T}}_2\right)\subset\left[\underset{\sim}{\mathbf{S}}_A, \underset{\sim}{\mathbf{S}}_A^{-1}, \underset{\sim}{\mathbf{A}}_A, \underset{\sim}{\mathbf{B}}_A, \underset{\sim}{\mathbf{B}}_A^{-1}, \underset{\sim}{\mathbf{S}}_B, \underset{\sim}{\mathbf{S}}_B^{-1}, \underset{\sim}{\mathbf{A}}_B, \underset{\sim}{\mathbf{B}}_B, \underset{\sim}{\mathbf{B}}_B^{-1}\right].$$

While the *symmetric product* and *A-bracket* operators are similar in nature, the *antisymmetric product* is quite different. Most significant is that there are no conditions on $\mathbf{A}\in\mathcal{S}$ which will insure the invertibility of $\underset{\sim}{\mathbf{A}}_A$. In fact, the linear operator $\underset{\sim}{\mathbf{A}}_A$ (by definition) has a non-trivial *null space* consisting of all second order tensors which commute with it[24]. This is formally expressed as

$$\underset{\sim}{\mathbf{A}}_A\cdot\mathbf{X}=\mathbf{0} \;\Leftrightarrow\; \mathbf{X}\in\mathcal{C}(\mathbf{A})\equiv\{\mathbf{X}:\mathbf{AX}=\mathbf{XA}\}\subset\mathcal{T}, \tag{A.21}$$

in terms of the *commutative tensor subspace* $\mathcal{C}(\mathbf{A})$ of $\mathbf{A}\in\mathcal{S}$. In light of the matrix forms (A.8) and

[24] Consideration of the tensor equation

$$2\underset{\sim}{\mathbf{A}}_A\cdot\mathbf{X} = \mathbf{AX}-\mathbf{XA} = \mathbf{C}$$

for any $\mathbf{A}\in\mathcal{T}$ is the specific focus of Guo *et al.* (1992).

$(A.13)_2$, examination of the homogeneous linear system

$$\underset{\sim}{\mathbf{A}}_A \mathbf{X} = \mathbf{0} \quad \Leftrightarrow \quad [\![\underset{\sim}{\mathbf{A}}_A \mathbf{X}]\!]_{\mathbf{A}_\alpha} = [\![\underset{\sim}{\mathbf{A}}_A]\!]_{\mathbf{A}_\alpha} [\![\mathbf{X}]\!]_{\mathbf{A}_\alpha} = [\![\mathbf{0}\text{'s}]\!]$$

reveals that this linear subspace $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{T}$ is:

(i) three (3) dimensional and spanned by the symmetric tensor set

$$\{\mathbf{A}_\alpha\}_{\alpha=1}^{3} = [(\hat{A}_1 \otimes \hat{A}_1), (\hat{A}_2 \otimes \hat{A}_2), (\hat{A}_3 \otimes \hat{A}_3)]$$

whenever $\mathbf{A} \in \mathcal{S}$ has *distinct* eigenvalues, that is

$$(A_2 - A_3)^2 (A_3 - A_1)^2 (A_1 - A_2)^2 > 0 \, ; \tag{A.22}$$

(ii) five (5) dimensional and spanned by the tensor set

$$\{\mathbf{A}_\alpha\}_{\alpha=1}^{5} = [(\hat{A}_1 \otimes \hat{A}_1), (\hat{A}_2 \otimes \hat{A}_2), (\hat{A}_3 \otimes \hat{A}_3), (\hat{A}_2 \otimes \hat{A}_3), (\hat{A}_3 \otimes \hat{A}_2)]$$

whenever $\mathbf{A} \in \mathcal{S}$ has only two *distinct* eigenvalues with

$$A_1 \neq A_2 = A_3 \quad \Rightarrow \quad (A_2 - A_3)^2 + (A_3 - A_1)^2 + (A_1 - A_2)^2 > 0 \, ; \tag{A.23}$$

(iii) nine (9) dimensional, consisting of the entire space $\mathcal{T}$ whenever $\mathbf{A} \in \mathcal{S}$ is *isotropic*, that is

$$A_1 = A_2 = A_3 \quad \Rightarrow \quad (A_2 - A_3)^2 + (A_3 - A_1)^2 + (A_1 - A_2)^2 = 0 \, . \tag{A.24}$$

Associated with this *commutative tensor subspace* $\mathcal{C}(\mathbf{A})$ for any given $\mathbf{A} \in \mathcal{S}$, is its *complimentary* (normal) subspace defined by

$$\mathcal{C}^*(\mathbf{A}) = outer[\mathcal{C}(\mathbf{A})] \equiv \{\mathbf{Y} : \mathbf{Y} \bullet \mathbf{X} = 0 \ \forall \ \mathbf{X} \in \mathcal{C}(\mathbf{A})\} \subset \mathcal{T}. \tag{A.25}$$

Depending on the multiplicity of distinct $\mathbf{A}$-eigenvalues, this linear subspace has dimension of either six (6=9-3), four (4=9-5), or zero (0=9-9). Corresponding to the non-trivial cases (i) and (ii) above, it is clearly spanned by tensor subsets

$$\text{(i)} \quad \{\mathbf{A}_\alpha\}_{\alpha=4}^{9} = [(\hat{A}_2 \otimes \hat{A}_3), (\hat{A}_3 \otimes \hat{A}_2), (\hat{A}_3 \otimes \hat{A}_1), (\hat{A}_1 \otimes \hat{A}_3), (\hat{A}_1 \otimes \hat{A}_2), (\hat{A}_2 \otimes \hat{A}_1)] \, ,$$

$$\text{(ii)} \quad \{\mathbf{A}_\alpha\}_{\alpha=6}^{9} = [(\hat{A}_3 \otimes \hat{A}_1), (\hat{A}_1 \otimes \hat{A}_3), (\hat{A}_1 \otimes \hat{A}_2), (\hat{A}_2 \otimes \hat{A}_1)] \, . \tag{A.26}$$

For given $\mathbf{A} \in \mathcal{S}$, it is interesting to note that $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}^*(\mathbf{A})$ allow for a *direct sum decomposition* of the tensor space $\mathcal{T}$. This is formally expressed as

$$\mathcal{T} = \mathcal{C}(\mathbf{A}) \oplus \mathcal{C}^*(\mathbf{A}) \, ,$$

which is symbolic of the fact that every $\mathbf{X} \in \mathcal{T}$ admits a unique decomposition of the form

$$\mathbf{X} = [\mathbf{X}]_{\mathcal{C}(\mathbf{A})} + [\mathbf{X}]_{\mathcal{C}^*(\mathbf{A})} \; ; \quad \begin{cases} [\mathbf{X}]_{\mathcal{C}(\mathbf{A})} \in \mathcal{C}(\mathbf{A}) \, , \\ [\mathbf{X}]_{\mathcal{C}^*(\mathbf{A})} \in \mathcal{C}^*(\mathbf{A}) \, , \end{cases}$$

with (A.27)

$$\| \mathbf{X} \|^2 = \| [\mathbf{X}]_{\mathcal{C}(\mathbf{A})} \|^2 + \| [\mathbf{X}]_{\mathcal{C}^*(\mathbf{A})} \|^2 \, .$$

Moreover, each of these subspaces contains both the symmetric and antisymmetric components of each and every one of its elements as expressed by the relations

$$\mathbf{X} \in \mathcal{C}(\mathbf{A}) \quad \Leftrightarrow \quad \{[\mathbf{X}]_S, [\mathbf{X}]_A\} \subset \mathcal{C}(\mathbf{A}) \, ,$$

$$\mathbf{X} \in \mathcal{C}^*(\mathbf{A}) \quad \Leftrightarrow \quad \{[\mathbf{X}]_S, [\mathbf{X}]_A\} \subset \mathcal{C}^*(\mathbf{A}) \, .$$

For given $\mathbf{A} \in \mathcal{S}$, the above complimentary subspaces $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}^*(\mathbf{A})$ allow for a complete characterization of the mapping properties of the linear operator $\underset{\sim}{\mathbf{A}}_{\mathrm{A}}$. As noted in (A.21), the three (3), five (5) or nine (9) dimensional *commutative tensor subspace* $\mathcal{C}(\mathbf{A})$ forms the *null space* for the linear operator $\underset{\sim}{\mathbf{A}}_{\mathrm{A}}$ as expressed by

$$\underset{\sim}{\mathbf{A}}_{\mathrm{A}} : \mathcal{C}(\mathbf{A}) \rightarrow \{\mathbf{0}\}.$$

After reexamining the matrix form of the tensor mapping $\mathbf{Y} = \underset{\sim}{\mathbf{A}}_{\mathrm{A}} \mathbf{X}$, in light of the above established basic structure of these subspaces, it is equally clear that the complimentary *outer* space $\mathcal{C}^*(\mathbf{A})$ serves as its *range space* in the sense that

$$\underset{\sim}{\mathbf{A}}_{\mathrm{A}} : \mathcal{T} \rightarrow \mathcal{C}^*(\mathbf{A}).$$

Moreover, the restriction of $\underset{\sim}{\mathbf{A}}_{\mathrm{A}}$ to the linear subspace $\mathcal{C}^*(\mathbf{A})$,

$$\underset{\sim}{\mathbf{A}}_{\mathrm{A}} \big|_{\mathcal{C}^*(\mathbf{A})} : \mathcal{C}^*(\mathbf{A}) \rightarrow \mathcal{C}^*(\mathbf{A}),$$

is one-to-one. Thus, **for each and every $\mathbf{Y} \in \mathcal{C}^*(\mathbf{A})$, there exists one and only one $\mathbf{X} \in \mathcal{C}^*(\mathbf{A})$ such that**

$$\mathbf{Y} = \underset{\sim}{\mathbf{A}}_{\mathrm{A}} \mathbf{X}. \tag{A.28}$$

Put differently, for each $\mathbf{Y} \in \mathcal{C}^*(\mathbf{A})$, there exists a unique tensor solution $\mathbf{X} \in \mathcal{C}^*(\mathbf{A})$ to the equation $\mathbf{Y} = \underset{\sim}{\mathbf{A}}_{\mathrm{A}} \mathbf{X}$.

In view of these last considerations relating to the basic structure of the complimentary subspaces $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}^*(\mathbf{A})$, it is as easily established that **any fourth order tensor $\underset{\sim}{\mathbf{T}}$ which has an A-eigenbasis representation of the specific diagonal submatrix form**

$$\{\mathrm{T}_{\alpha\beta}\}_{9\times9} = [\![\underset{\sim}{\mathbf{T}}]\!]_{\mathrm{A}_\alpha} = \mathit{diagonal}\left\{[\![\mathbf{T}_0]\!]_{3\times3}, [\![\mathbf{T}_1]\!]_{2\times2}, [\![\mathbf{T}_2]\!]_{2\times2}, [\![\mathbf{T}_3]\!]_{2\times2}\right\} = \begin{pmatrix} \cdot&\cdot&\cdot&0&0&0&0&0&0 \\ \cdot&\cdot&\cdot&0&0&0&0&0&0 \\ \cdot&\cdot&\cdot&0&0&0&0&0&0 \\ 0&0&0&\cdot&\cdot&0&0&0&0 \\ 0&0&0&\cdot&\cdot&0&0&0&0 \\ 0&0&0&0&0&\cdot&\cdot&0&0 \\ 0&0&0&0&0&\cdot&\cdot&0&0 \\ 0&0&0&0&0&0&0&\cdot&\cdot \\ 0&0&0&0&0&0&0&\cdot&\cdot \end{pmatrix}$$

will preserve the linear subspaces $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}^*(\mathbf{A})$ in the sense that (A.29)[25]

$$\underset{\sim}{\mathbf{T}} : \mathcal{C}(\mathbf{A}) \rightarrow \mathcal{C}(\mathbf{A}) \quad \& \quad \underset{\sim}{\mathbf{T}} : \mathcal{C}^*(\mathbf{A}) \rightarrow \mathcal{C}^*(\mathbf{A}).$$

Moreover, $\underset{\sim}{\mathbf{T}}$ is a *symmetric* bilinear form on $\mathcal{T}$ if and only if each of these four submatrices is symmetric. In addition, if these submatrix components satisfy the conditions

$$\begin{aligned} & [\![\mathbf{T}_0]\!]_{3\times3} = \mathbf{0}, \\ [\mathrm{A}_2 = \mathrm{A}_3] \quad \Rightarrow \quad & [\![\mathbf{T}_1]\!]_{2\times2} = \mathbf{0}, \\ [\mathrm{A}_3 = \mathrm{A}_1] \quad \Rightarrow \quad & [\![\mathbf{T}_2]\!]_{2\times2} = \mathbf{0}, \\ [\mathrm{A}_1 = \mathrm{A}_2] \quad \Rightarrow \quad & [\![\mathbf{T}_3]\!]_{2\times2} = \mathbf{0}, \end{aligned}$$

(as does $\underset{\sim}{\mathbf{A}}_{\mathrm{A}}$) then it is assured that (A.30)

$$\underset{\sim}{\mathbf{T}} : \mathcal{C}(\mathbf{A}) \rightarrow \{\mathbf{0}\} \quad \Rightarrow \quad \underset{\sim}{\mathbf{T}} : \mathcal{T} \rightarrow \mathcal{C}^*(\mathbf{A}).$$

---

[25] Observe that the ‘Hookean’ operator

$$[\![\underset{\sim}{\mathbf{K}}]\!]_{\mathrm{A}_\alpha} = \mathit{diagonal}\left\{\frac{1}{\mathrm{E}}\begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix}, \frac{1}{2\mathrm{G}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2\mathrm{G}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2\mathrm{G}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\} \; ; \quad \mathrm{G} = \frac{\mathrm{E}}{2(1+\nu)},$$

which maps linearly elastic stresses into strains, is a fourth order tensor of this type.

Finally, if these submatrices satisfy the even more stringent requirements

$$\begin{aligned}&[\![\mathbf{T}_o]\!]_{3\times3} = \mathbf{0}\,,\\ &[\mathrm{A}_i = \mathrm{A}_j] \;\Rightarrow\; [\![\mathbf{T}_k]\!]_{2\times2} = \mathbf{0} \\ &[\mathrm{A}_i \neq \mathrm{A}_j] \;\Rightarrow\; det\left([\![\mathbf{T}_k]\!]_{2\times2}\right) \neq \mathbf{0}\end{aligned}\Bigg\}; \quad (i,j,k) = \begin{bmatrix}(2,3,1)\\(3,1,2)\\(1,2,3)\end{bmatrix},$$

then it is also guaranteed that (A.31)

$$\underset{\sim}{\mathbf{T}}\big|_{C^*(\mathbf{A})} : C^*(\mathbf{A}) \to C^*(\mathbf{A})$$

is one-to-one on $C^*(\mathbf{A})$ in the same sense as the operator $\underset{\sim}{\mathbf{A}}_{\mathrm{A}}$ above.

<u>The widget</u> (for any positive/negative definite $\mathbf{A} \in \mathcal{S}$)

For any $\mathbf{X} \in \mathcal{T}$ and <u>definite</u> $\mathbf{A} \in \mathcal{S}$, the obvious identity

$$\begin{aligned}&\underset{\sim}{\mathbf{S}}_{\mathrm{A}} \circ \underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1} \mathbf{X} = \mathbf{X}\,,\\ &\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right) + \left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right)\mathbf{A} = 2\mathbf{X}\,,\\ &\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right) - \mathbf{X} = \mathbf{X} - \left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right)\mathbf{A}\,,\end{aligned}$$

inspires the alternative definitions of the linear operator

$$\underset{\sim}{\mathbf{W}}_{\mathrm{A}} : \mathcal{T} \to \mathcal{T}$$

as

$$\boxed{\underset{\sim}{\mathbf{W}}_{\mathrm{A}}\mathbf{X} \equiv \left\{ \begin{array}{c}\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right) - \mathbf{X} \\ \mathbf{X} - \left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right)\mathbf{A}\end{array}\right.}\,. \tag{A.32}$$

This new fourth order tensor operator, to which is attached the mnemonic handle "widget," plays an important role in the development of the present theory. From this definition, the established property (A.15), and the development

$$\begin{aligned}\left[\underset{\sim}{\mathbf{W}}_{\mathrm{A}}\mathbf{X}\right]^{\mathrm{T}} &= \left[\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right) - \mathbf{X}\right]^{\mathrm{T}}\\ &= \left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}\right)^{\mathrm{T}}\mathbf{A} - \mathbf{X}^{\mathrm{T}}\\ &= \left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}^{\mathrm{T}}\right)\mathbf{A} - \mathbf{X}^{\mathrm{T}}\\ &= -\left[\mathbf{X}^{\mathrm{T}} - \left(\underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\mathbf{X}^{\mathrm{T}}\right)\mathbf{A}\right]\end{aligned}$$

$$\boxed{\left[\underset{\sim}{\mathbf{W}}_{\mathrm{A}}\mathbf{X}\right]^{\mathrm{T}} = -\left[\underset{\sim}{\mathbf{W}}_{\mathrm{A}}\mathbf{X}^{\mathrm{T}}\right]}\,, \tag{A.33}$$

it is evident that the *widget*, like $\underset{\sim}{\mathbf{A}}_{\mathrm{A}}$, linearly maps symmetric/antisymmetric tensors into antisymmetric/symmetric tensors. With the aid of (A.4)$_5$ and the associated identity

$$\underset{\sim}{\mathbf{S}}_{\mathrm{A}^{-1}} \circ \underset{\sim}{\mathbf{B}}_{\mathrm{A}} = \underset{\sim}{\mathbf{S}}_{\mathrm{A}} \;\Leftrightarrow\; \underset{\sim}{\mathbf{S}}_{\mathrm{A}^{-1}} = \underset{\sim}{\mathbf{S}}_{\mathrm{A}} \circ \underset{\sim}{\mathbf{B}}_{\mathrm{A}}^{-1} \;\Leftrightarrow\; \underset{\sim}{\mathbf{S}}_{\mathrm{A}^{-1}}^{-1} = \underset{\sim}{\mathbf{B}}_{\mathrm{A}} \circ \underset{\sim}{\mathbf{S}}_{\mathrm{A}}^{-1}\,,$$

the relations (A.32), (A.1)$_3$, and the development

$$
\begin{aligned}
\underset{\sim}{\mathbf{W}}_{A^{-1}}\cdot\mathbf{X} &= \mathbf{X} - \left(\underset{\sim}{\mathbf{S}}_{A^{-1}}^{-1}\cdot\mathbf{X}\right)\mathbf{A}^{-1} \\
&= \mathbf{X} - \left[\left(\underset{\sim}{\mathbf{B}}_{A}\circ\underset{\sim}{\mathbf{S}}_{A}^{-1}\right)\cdot\mathbf{X}\right]\mathbf{A}^{-1} \\
&= \mathbf{X} - \left[\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\cdot\mathbf{X}\right)\mathbf{A}\right]\mathbf{A}^{-1} \\
&= -\left[\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\cdot\mathbf{X}\right) - \mathbf{X}\right] \\
\underset{\sim}{\mathbf{W}}_{A^{-1}}\cdot\mathbf{X} &= -\left(\underset{\sim}{\mathbf{W}}_{A}\cdot\mathbf{X}\right),
\end{aligned}
$$

it then follows that the *widget* satisfies the interesting identity

$$\underset{\sim}{\mathbf{W}}_{A^{-1}} = -\underset{\sim}{\mathbf{W}}_{A}\ .$$

Yet another interesting result, namely

$$\underset{\sim}{\mathbf{S}}_{A^{-1}}\circ\underset{\sim}{\mathbf{W}}_{A} = -\underset{\sim}{\mathsf{A}}_{A^{-1}} \quad\Leftrightarrow\quad \underset{\sim}{\mathbf{W}}_{A} = -\underset{\sim}{\mathbf{S}}_{A^{-1}}^{-1}\circ\underset{\sim}{\mathsf{A}}_{A^{-1}}\ ,$$

follows as a consequence of the expansion

$$
\begin{aligned}
\left(\underset{\sim}{\mathbf{S}}_{A^{-1}}\circ\underset{\sim}{\mathbf{W}}_{A}\right)\mathbf{X} &= \tfrac{1}{2}\left[\mathbf{A}^{-1}\left(\underset{\sim}{\mathbf{W}}_{A}\cdot\mathbf{X}\right) + \left(\underset{\sim}{\mathbf{W}}_{A}\cdot\mathbf{X}\right)\mathbf{A}^{-1}\right] \\
&= \tfrac{1}{2}\left\{\mathbf{A}^{-1}\left[\mathbf{A}\left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\cdot\mathbf{X}\right) - \mathbf{X}\right] + \left[\mathbf{X} - \left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\cdot\mathbf{X}\right)\mathbf{A}\right]\mathbf{A}^{-1}\right\} \\
&= \tfrac{1}{2}\left[-\mathbf{A}^{-1}\mathbf{X} + \mathbf{X}\mathbf{A}^{-1} + \left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\cdot\mathbf{X}\right) - \left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\cdot\mathbf{X}\right)\right] \\
\left(\underset{\sim}{\mathbf{S}}_{A^{-1}}\circ\underset{\sim}{\mathbf{W}}_{A}\right)\mathbf{X} &= -\underset{\sim}{\mathsf{A}}_{A^{-1}}\cdot\mathbf{X}\,.
\end{aligned}
$$

These last two results are now combined to establish the critical identity

$$\boxed{\underset{\sim}{\mathbf{W}}_{A} = -\underset{\sim}{\mathbf{S}}_{A^{-1}}^{-1}\circ\underset{\sim}{\mathsf{A}}_{A^{-1}} = -\underset{\sim}{\mathbf{W}}_{A^{-1}} = \underset{\sim}{\mathbf{S}}_{A}^{-1}\circ\underset{\sim}{\mathsf{A}}_{A}} \;;\quad \left[(\mathbf{A}^{-1})^{-1} = \mathbf{A}\right]. \tag{A.34}$$

In view of this, the symmetry and commutative properties (A.2), (A.14)$_2$, and (A.20), and the development

$$\underset{\sim}{\mathbf{W}}_{A}^{T} = \left(\underset{\sim}{\mathbf{S}}_{A}^{-1}\circ\underset{\sim}{\mathsf{A}}_{A}\right)^{T} = \underset{\sim}{\mathsf{A}}_{A}^{T}\circ\underset{\sim}{\mathbf{S}}_{A}^{-T} = \underset{\sim}{\mathsf{A}}_{A}\circ\underset{\sim}{\mathbf{S}}_{A}^{-1} = \underset{\sim}{\mathbf{S}}_{A}^{-1}\circ\underset{\sim}{\mathsf{A}}_{A} = \underset{\sim}{\mathbf{W}}_{A}\ ,$$

the *widget* is also established as a symmetric bilinear form on $\mathcal{T}$ as expressed by

$$\boxed{\mathbf{X}\bullet\left(\underset{\sim}{\mathbf{W}}_{A}\cdot\mathbf{Y}\right) = \mathbf{Y}\bullet\left(\underset{\sim}{\mathbf{W}}_{A}\cdot\mathbf{X}\right) \;;\quad (\mathbf{X},\mathbf{Y})\subset\mathcal{T}}\ . \tag{A.35}$$

Alternatively, these general properties could have been established by examining the properties of its **A**-eigenbasis representation matrix. In view of (A.34) and the matrix forms (A.13)$_2$ and (A.16)$_2$, this is now easily obtained as

$$
\begin{aligned}
\left[\!\left[\underset{\sim}{\mathbf{W}}_{A}\right]\!\right]_{\mathbf{A}_\alpha} &= \left[\!\left[\underset{\sim}{\mathbf{S}}_{A}^{-1}\right]\!\right]_{\mathbf{A}_\alpha}\left[\!\left[\underset{\sim}{\mathsf{A}}_{A}\right]\!\right]_{\mathbf{A}_\alpha} \\
&= diagonal\left[0,0,0,\left(\frac{A_2-A_3}{A_2+A_3}\right),-\left(\frac{A_2-A_3}{A_2+A_3}\right),\left(\frac{A_3-A_1}{A_3+A_1}\right),-\left(\frac{A_3-A_1}{A_3+A_1}\right),\left(\frac{A_1-A_2}{A_1+A_2}\right),-\left(\frac{A_1-A_2}{A_1+A_2}\right)\right] \\
\left[\!\left[\underset{\sim}{\mathbf{W}}_{A}\right]\!\right]_{\mathbf{A}_\alpha} &= diagonal\left[0,0,0,\left(\frac{A_2-A_3}{A_2+A_3}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix},\left(\frac{A_3-A_1}{A_3+A_1}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix},\left(\frac{A_1-A_2}{A_1+A_2}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right].
\end{aligned} \tag{A.36}
$$

After verifying that this representation matrix satisfies the requirements (A.31), the *widget* is also confirmed to have $\mathcal{C}(\mathbf{A})$ as its *null* space

$$\underset{\sim}{\mathbf{W}}_{A} : \mathcal{C}(\mathbf{A}) \to \{\mathbf{0}\}\ ,$$

$\mathcal{C}^{*}(\mathbf{A})$ as its *range* space

$$\underset{\sim}{\mathbb{W}}_A : \mathcal{T} \to C^*(\mathbf{A}),$$

and has the one-to-one restriction

$$\underset{\sim}{\mathbb{W}}_A \big|_{C^*(\mathbf{A})} : C^*(\mathbf{A}) \to C^*(\mathbf{A}).$$

Finally, it is a simple exercise to establish the following special eigenbasis expansions for symmetric and antisymmetric tensor arguments:

$$\begin{aligned}
\mathbf{X} = \mathbf{X}^T \quad \Rightarrow \quad \underset{\sim}{\mathbb{W}}_A \mathbf{X} = & \left(\frac{A_2 - A_3}{A_2 + A_3}\right)(\hat{A}_2 \bullet \mathbf{X} \hat{A}_3)[(\hat{A}_2 \otimes \hat{A}_3) - (\hat{A}_3 \otimes \hat{A}_2)] \\
& + \left(\frac{A_3 - A_1}{A_3 + A_1}\right)(\hat{A}_3 \bullet \mathbf{X} \hat{A}_1)[(\hat{A}_3 \otimes \hat{A}_1) - (\hat{A}_1 \otimes \hat{A}_3)] \\
& + \left(\frac{A_1 - A_2}{A_1 + A_2}\right)(\hat{A}_1 \bullet \mathbf{X} \hat{A}_2)[(\hat{A}_1 \otimes \hat{A}_2) - (\hat{A}_2 \otimes \hat{A}_1)], \\
\mathbf{X} = -\mathbf{X}^T \quad \Rightarrow \quad \underset{\sim}{\mathbb{W}}_A \mathbf{X} = & \left(\frac{A_2 - A_3}{A_2 + A_3}\right)(\hat{A}_2 \bullet \mathbf{X} \hat{A}_3)[(\hat{A}_2 \otimes \hat{A}_3) + (\hat{A}_3 \otimes \hat{A}_2)] \\
& + \left(\frac{A_3 - A_1}{A_3 + A_1}\right)(\hat{A}_3 \bullet \mathbf{X} \hat{A}_1)[(\hat{A}_3 \otimes \hat{A}_1) + (\hat{A}_1 \otimes \hat{A}_3)] \\
& + \left(\frac{A_1 - A_2}{A_1 + A_2}\right)(\hat{A}_1 \bullet \mathbf{X} \hat{A}_2)[(\hat{A}_1 \otimes \hat{A}_2) + (\hat{A}_2 \otimes \hat{A}_1)].
\end{aligned} \tag{A.37}$$

Special forms [involving the elastic strain measures $\mathbf{v} = \mathbf{u}^{-1} = \sqrt{\mathbf{b}} = exp(\mathbf{a})$]

The specific applications in this work involve a positive definite, symmetric elastic stretch tensor $\mathbf{v} \in \mathcal{S}$ and the related symmetric strain measures

$$\mathbf{u} = \mathbf{v}^{-1} \;;\quad \mathbf{b} = \mathbf{v}^2 \;;\quad \mathbf{a} = ln(\mathbf{v}). \tag{A.38}$$

In terms of the value/vector eigenpairs $\{a_k, \hat{a}_k\}_{k=1}^3$ for the *log strain* tensor

$$\mathbf{a} = ln(\mathbf{v}) = a_1(\hat{a}_1 \otimes \hat{a}_1) + a_2(\hat{a}_2 \otimes \hat{a}_2) + a_3(\hat{a}_3 \otimes \hat{a}_3),$$

it is clear that

$$\begin{aligned}
\mathbf{v} = exp(\mathbf{a}) &= e^{a_1}(\hat{a}_1 \otimes \hat{a}_1) + e^{a_2}(\hat{a}_2 \otimes \hat{a}_2) + e^{a_3}(\hat{a}_3 \otimes \hat{a}_3), \\
\mathbf{u} = exp(-\mathbf{a}) &= e^{-a_1}(\hat{a}_1 \otimes \hat{a}_1) + e^{-a_2}(\hat{a}_2 \otimes \hat{a}_2) + e^{-a_3}(\hat{a}_3 \otimes \hat{a}_3), \\
\mathbf{b} = exp(2\mathbf{a}) &= e^{2a_1}(\hat{a}_1 \otimes \hat{a}_1) + e^{2a_2}(\hat{a}_2 \otimes \hat{a}_2) + e^{2a_3}(\hat{a}_3 \otimes \hat{a}_3).
\end{aligned}$$

Throughout this work, the *widget* operator $\underset{\sim}{\mathbb{W}}_v$ plays an important roll. Before examining its eigenbasis matrix representation (A.36), it is interesting to observe that the $\mathbf{v}$-eigenvalues satisfy the relations

$$\begin{aligned}
\mu_1 &\equiv a_2 - a_3 = ln(v_2) - ln(v_3) = ln(v_2/v_3) \quad \Leftrightarrow \quad v_2/v_3 = e^{\mu_1}, \\
\mu_2 &\equiv a_3 - a_1 = ln(v_3) - ln(v_1) = ln(v_3/v_1) \quad \Leftrightarrow \quad v_3/v_1 = e^{\mu_2}, \\
\mu_3 &\equiv a_1 - a_2 = ln(v_1) - ln(v_2) = ln(v_1/v_2) \quad \Leftrightarrow \quad v_1/v_2 = e^{\mu_3},
\end{aligned} \tag{A.39}$$

in terms of the *principal differences* of $\mathbf{a} = ln(\mathbf{v})$. This, in turn, leads to the multiple identities

$$\begin{aligned}
\frac{v_2 - v_3}{v_2 + v_3} &= \frac{v_2/v_3 - 1}{v_2/v_3 + 1} \\
&= \frac{e^{\mu_1} - 1}{e^{\mu_1} + 1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{\mu_1}-1}{e^{\mu_1}+1}\cdot\left(\frac{\frac{1}{2}e^{-\mu_1/2}}{\frac{1}{2}e^{-\mu_1/2}}\right) \\
&= \frac{\frac{1}{2}\left[e^{\mu_1/2}-e^{-\mu_1/2}\right]}{\frac{1}{2}\left[e^{\mu_1/2}+e^{-\mu_1/2}\right]} = \frac{\frac{1}{2}\left[e^{\mu_1/2}-e^{-\mu_1/2}\right]}{\frac{1}{2}\left[e^{\mu_1/2}+e^{-\mu_1/2}\right]}\cdot\left(\frac{e^{\mu_1/2}-e^{-\mu_1/2}}{e^{\mu_1/2}-e^{-\mu_1/2}}\right) \\
&= \frac{sinh\left(\frac{1}{2}\mu_1\right)}{cosh\left(\frac{1}{2}\mu_1\right)} = \frac{\frac{1}{2}\left[e^{\mu_1}+e^{-\mu_1}\right]-1}{\frac{1}{2}\left[e^{\mu_1}-e^{-\mu_1}\right]} \\
\frac{v_2-v_3}{v_2+v_3} &= tanh\left(\tfrac{1}{2}\mu_1\right) = \frac{cosh(\mu_1)-1}{sinh(\mu_1)} = \frac{1}{tanh(\mu_1)}-\frac{1}{sinh(\mu_1)},
\end{aligned}
$$

and in identical fashion,

$$
\frac{v_i-v_j}{v_i+v_j} = \left\langle \begin{matrix} tanh\left(\frac{1}{2}\mu_k\right) \\ \dfrac{cosh(\mu_k)-1}{sinh(\mu_k)} \\ \dfrac{1}{tanh(\mu_k)}-\dfrac{1}{sinh(\mu_k)} \end{matrix} \right\rangle ; \quad (i,j,k)\in\begin{bmatrix}(1,2,3)\\(2,3,1)\\(3,1,2)\end{bmatrix}, \tag{A.40}
$$

These allow for a restatement of the matrix representation (A.36) and the expansions (A.37) in the specialized forms:

$$
\left[\!\left[\underset{\sim}{\mathbf{W}}_v \right]\!\right]_{\mathbf{A}_\alpha} = diagonal\left[0,0,0, tanh\left(\tfrac{1}{2}\mu_1\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}, tanh\left(\tfrac{1}{2}\mu_2\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}, tanh\left(\tfrac{1}{2}\mu_3\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right], \tag{A.41}
$$

$$
\begin{aligned}
\mathbf{X}=\mathbf{X}^{\mathrm{T}} \;\Rightarrow\; \underset{\sim}{\mathbf{W}}_v\mathbf{X} &= tanh\left(\tfrac{1}{2}\mu_1\right)(\hat{a}_2\bullet\mathbf{X}\hat{a}_3)[(\hat{a}_2\otimes\hat{a}_3)-(\hat{a}_3\otimes\hat{a}_2)] \\
&+ tanh\left(\tfrac{1}{2}\mu_2\right)(\hat{a}_3\bullet\mathbf{X}\hat{a}_1)[(\hat{a}_3\otimes\hat{a}_1)-(\hat{a}_1\otimes\hat{a}_3)] \\
&+ tanh\left(\tfrac{1}{2}\mu_3\right)(\hat{a}_1\bullet\mathbf{X}\hat{a}_2)[(\hat{a}_1\otimes\hat{a}_2)-(\hat{a}_2\otimes\hat{a}_1)], \\
\mathbf{X}=-\mathbf{X}^{\mathrm{T}} \;\Rightarrow\; \underset{\sim}{\mathbf{W}}_v\mathbf{X} &= tanh\left(\tfrac{1}{2}\mu_1\right)(\hat{a}_2\bullet\mathbf{X}\hat{a}_3)[(\hat{a}_2\otimes\hat{a}_3)+(\hat{a}_3\otimes\hat{a}_2)] \\
&+ tanh\left(\tfrac{1}{2}\mu_2\right)(\hat{a}_3\bullet\mathbf{X}\hat{a}_1)[(\hat{a}_3\otimes\hat{a}_1)+(\hat{a}_1\otimes\hat{a}_3)] \\
&+ tanh\left(\tfrac{1}{2}\mu_3\right)(\hat{a}_1\bullet\mathbf{X}\hat{a}_2)[(\hat{a}_1\otimes\hat{a}_2)+(\hat{a}_2\otimes\hat{a}_1)].
\end{aligned} \tag{A.42}
$$

In Section 3, an antisymmetric solution $\mathbf{X}=-\mathbf{X}^{\mathrm{T}}$ to the tensor equation

$$
\mathbf{Y}=\mathbf{Y}^{\mathrm{T}}=\underset{\sim}{\mathbf{A}}_a\mathbf{X}\in\mathcal{S}
$$

is required. As noted in (A.28), there exists a unique solution within the tensor subspace $\mathcal{C}^*(\mathbf{a})$ provided that $\mathbf{Y}=\mathbf{Y}^{\mathrm{T}}\in\mathcal{C}^*(\mathbf{a})$. After rewriting the matrix representation $(A.13)_2$ in terms of the principal differences (A.39),

$$
\left[\!\left[\underset{\sim}{\mathbf{A}}_a \right]\!\right]_{\mathbf{A}_\alpha} = \frac{1}{2}diagonal\left[0,0,0,\mu_1\begin{pmatrix}1&0\\0&-1\end{pmatrix},\mu_2\begin{pmatrix}1&0\\0&-1\end{pmatrix},\mu_3\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right], \tag{A.43}
$$

the useful eigenbasis expansion

$$
\begin{aligned}
\mathbf{X}=-\mathbf{X}^{\mathrm{T}} \;\Rightarrow\; \underset{\sim}{\mathbf{A}}_a\mathbf{X} = \tfrac{1}{2}\{&\mu_1(\hat{a}_2\bullet\mathbf{X}\hat{a}_3)[(\hat{a}_2\otimes\hat{a}_3)+(\hat{a}_3\otimes\hat{a}_2)] \\
&+ \mu_2(\hat{a}_3\bullet\mathbf{X}\hat{a}_1)[(\hat{a}_3\otimes\hat{a}_1)+(\hat{a}_1\otimes\hat{a}_3)] \\
&+ \mu_3(\hat{a}_1\bullet\mathbf{X}\hat{a}_2)[(\hat{a}_1\otimes\hat{a}_2)+(\hat{a}_2\otimes\hat{a}_1)]\}
\end{aligned} \tag{A.44}
$$

is readily obtained.

Near the end of Section 3, there is an important identity involving the compound operator $(2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v)$ concerning its action on the set of symmetric tensors $\mathcal{S}$. In view of the established properties of each of these operators, it is immediately clear that it is a symmetric bilinear form on $\mathcal{T}$, mapping symmetric tensors into symmetric tensors. With the matrix forms (A.41) and (A.43), its **a**-eigenbasis matrix representation

$$[\![2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v]\!]_{\mathbf{A}_\alpha} = 2[\![\underset{\sim}{\mathbf{A}}_a]\!]_{\mathbf{A}_\alpha}[\![\underset{\sim}{\mathbf{W}}_v]\!]_{\mathbf{A}_\alpha}$$

$$[\![2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v]\!]_{\mathbf{A}_\alpha} = diagonal\left[0,0,0,\mu_1 tanh\left(\tfrac{1}{2}\mu_1\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mu_2 tanh\left(\tfrac{1}{2}\mu_2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mu_3 tanh\left(\tfrac{1}{2}\mu_3\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

is readily obtained. The specific form of this matrix is not only consistent with the above stated observations but, in light of the conditions (A.31), also serves to establish that $(2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v)$ has $\mathcal{C}(\mathbf{A})$ as its *null* space, and that it is one-to-one (and therefore invertible) when restricted to its *range* space $\mathcal{C}^*(\mathbf{a})$. In addition, this representation matrix gives rise to the useful **a**-eigenbasis expansion

$$\begin{aligned} \mathbf{X} = \mathbf{X}^T \Rightarrow (2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v)\mathbf{X} = {} & \mu_1 tanh\left(\tfrac{1}{2}\mu_1\right)(\hat{a}_2 \bullet \mathbf{X}\hat{a}_3)[(\hat{a}_2 \otimes \hat{a}_3) + (\hat{a}_3 \otimes \hat{a}_2)] \\ & + \mu_2 tanh\left(\tfrac{1}{2}\mu_2\right)(\hat{a}_3 \bullet \mathbf{X}\hat{a}_1)[(\hat{a}_3 \otimes \hat{a}_1) + (\hat{a}_1 \otimes \hat{a}_3)] \\ & + \mu_3 tanh\left(\tfrac{1}{2}\mu_3\right)(\hat{a}_1 \bullet \mathbf{X}\hat{a}_2)[(\hat{a}_1 \otimes \hat{a}_2) + (\hat{a}_2 \otimes \hat{a}_1)] \end{aligned} \tag{A.45}$$

for symmetric arguments $\mathbf{X} \in \mathcal{S}$. It shall also prove worthwhile to note that the scalar coefficients in this expression conform to the identity

$$\mu_k tanh\left(\tfrac{1}{2}\mu_k\right) = \frac{\mu_k}{tanh(\mu_k)} - \frac{\mu_k}{sinh(\mu_k)} \ ; \quad k = 1,2,3, \tag{A.46}$$

as a consequence of (A.40).

The $\mathbf{H}$-operators (and other 4th order tensor transformations on $\mathcal{S}$)

In the Appendix to Dashner (1990), the fourth order (double) tensors $\underset{\sim}{\mathbf{S}}_A$, $\underset{\sim}{\mathbf{B}}_A$, and $(\partial\mathbf{A}/\partial\mathbf{B})$ for $\mathbf{A} \in \mathcal{S}$ and $\mathbf{B} = exp(2\mathbf{A})$ were examined. As each is known to map symmetric tensors into symmetric tensors, their properties as linear transformations on the six (6) dimensional symmetric tensor subspace

$$\underset{\sim}{\mathbf{T}} : \mathcal{S} \rightarrow \mathcal{S},$$

can be, and were, developed by considering their respective matrix representations relative to the symmetric orthonormal tensor basis $\{{}^s\mathbf{A}_\kappa\}_{\kappa=1}^6 \subset \mathcal{S}$ defined by

$$\begin{aligned} {}^s\mathbf{A}_k &\equiv \hat{A}_k \otimes \hat{A}_k \ ; \quad k = 1,2,3, \\ {}^s\mathbf{A}_4 &\equiv \frac{1}{\sqrt{2}}\left[(\hat{A}_2 \otimes \hat{A}_3) + (\hat{A}_3 \otimes \hat{A}_2)\right], \\ {}^s\mathbf{A}_5 &\equiv \frac{1}{\sqrt{2}}\left[(\hat{A}_3 \otimes \hat{A}_1) + (\hat{A}_1 \otimes \hat{A}_3)\right], \\ {}^s\mathbf{A}_6 &\equiv \frac{1}{\sqrt{2}}\left[(\hat{A}_1 \otimes \hat{A}_2) + (\hat{A}_2 \otimes \hat{A}_1)\right], \end{aligned} \tag{A.47}$$

with

$${}^s\mathbf{A}_\mu \bullet {}^s\mathbf{A}_\nu = \delta_{\mu\nu} = \begin{cases} 1; & \text{whenever } \mu = \nu, \\ 0; & \text{whenever } \mu \neq \nu. \end{cases} \tag{A.48}$$

After confirming that any $\mathbf{X} \in \mathcal{S}$ has the unique column vector representation

$$\mathbf{X} = \sum_{i=1}^{3} \sum_{j=1}^{3} X_{ij}\left(\hat{A}_i \otimes \hat{A}_j\right)$$
$$\mathbf{X} = X_{11}\,{}^s\mathbf{A}_1 + X_{22}\,{}^s\mathbf{A}_2 + X_{33}\,{}^s\mathbf{A}_3 + \sqrt{2}X_{23}\,{}^s\mathbf{A}_4 + \sqrt{2}X_{31}\,{}^s\mathbf{A}_5 + \sqrt{2}X_{12}\,{}^s\mathbf{A}_6,$$
$$\Rightarrow \quad [\![\mathbf{X}]\!]_{{}^s\mathbf{A}_\alpha} = column\left(X_{11}, X_{22}, X_{33}, \sqrt{2}X_{23}, \sqrt{2}X_{31}, \sqrt{2}X_{12}\right), \qquad (A.49)$$

each of these operators is confirmed to have the dyadic representation

$$\underset{\sim}{\mathbf{T}} = \sum_{\alpha=1}^{6} \sum_{\beta=1}^{6} T^s_{\alpha\beta}\left({}^s\mathbf{A}_\alpha \otimes {}^s\mathbf{A}_\beta\right),$$

expressed in terms of its scalar representation matrix elements (A.50)

$$T^s_{\alpha\beta} = {}^s\mathbf{A}_\alpha \bullet \left(\underset{\sim}{\mathbf{T}}\,{}^s\mathbf{A}_\beta\right) \; ; \quad \alpha, \beta = 1, 2, \ldots, 6.$$

Consistent with the $\{\mathbf{A}_\alpha\}_{\alpha=1}^{9}$ basis representations $(A.13)_{1,3}$, it was shown that

$$\left\{S^s_{\alpha\beta}\right\}_{6x6} = [\![\underset{\sim}{\mathbf{S}}_A]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[A_1, A_2, A_3, \tfrac{1}{2}(A_2 + A_3), \tfrac{1}{2}(A_3 + A_1), \tfrac{1}{2}(A_1 + A_2)\right],$$
$$\left\{B^s_{\alpha\beta}\right\}_{6x6} = [\![\underset{\sim}{\mathbf{B}}_A]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[A_1^2, A_2^2, A_3^2, A_2A_3, A_3A_1, A_1A_2\right],$$

and that

$$\left\{(\partial A/\partial B)^s_{\alpha\beta}\right\}_{6x6} = [\![\partial \mathbf{A}/\partial \mathbf{B}]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[\frac{1}{2B_1}, \frac{1}{2B_2}, \frac{1}{2B_3}, \frac{A_2 - A_3}{B_2 - B_3}, \frac{A_3 - A_1}{B_3 - B_1}, \frac{A_1 - A_2}{B_1 - B_2}\right],$$

expressed in terms of the respective $\mathbf{A}$ and $\mathbf{B} = exp(2\mathbf{A})$ eigenvalues.

These results have direct application to the double tensors

$$\underset{\sim}{\mathbf{H}}_k : \mathcal{S} \to \mathcal{S} \; ; \quad k = 1, 2$$

defined as

$$\underset{\sim}{\mathbf{H}}_1 \equiv 2[\partial \mathbf{a}/\partial \mathbf{b}] \circ \underset{\sim}{\mathbf{S}}_b \quad \& \quad \underset{\sim}{\mathbf{H}}_2 \equiv 2[\partial \mathbf{a}/\partial \mathbf{b}] \circ \underset{\sim}{\mathbf{B}}_v$$

in terms of the elastic strain measures (A.38). In view of the above diagonal matrix forms, it immediately follows that these $\mathbf{H}$ tensors also have diagonal representations. After making the appropriate eigenvalue substitutions, the forms

$$[\![\underset{\sim}{\mathbf{H}}_1]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[1, 1, 1, \frac{b_2 + b_3}{b_2 - b_3}(a_2 - a_3), \frac{b_3 + b_1}{b_3 - b_1}(a_3 - a_1), \frac{b_1 + b_2}{b_1 - b_2}(a_1 - a_2)\right],$$
$$[\![\underset{\sim}{\mathbf{H}}_2]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[1, 1, 1, 2v_2v_3\left(\frac{a_2 - a_3}{b_2 - b_3}\right), 2v_3v_1\left(\frac{a_3 - a_1}{b_3 - b_1}\right), 2v_1v_2\left(\frac{a_1 - a_2}{b_1 - b_2}\right)\right],$$

are easily verified. By exploiting the eigenvalue relations (A.39), and the algebraic developments

$$\frac{b_2 + b_3}{b_2 - b_3}(a_2 - a_3) = \left(\frac{b_2/b_3 + 1}{b_2/b_3 - 1}\right)\mu_1$$
$$= \left(\frac{e^{2\mu_1} + 1}{e^{2\mu_1} - 1}\right)\mu_1 \; ; \quad \left[b_2/b_3 = (v_2/v_3)^2 = e^{2\mu_1}\right]$$

$$= \mu_1 \cdot \left(\frac{e^{2\mu_1}+1}{e^{2\mu_1}-1}\right)\left(\frac{\frac{1}{2}e^{-\mu_1}}{\frac{1}{2}e^{-\mu_1}}\right) = \mu_1 \cdot \left(\frac{\frac{1}{2}\left[e^{\mu_1}+e^{-\mu_1}\right]}{\frac{1}{2}\left[e^{\mu_1}-e^{-\mu_1}\right]}\right) = \mu_1 \cdot \left[\frac{cosh(\mu_1)}{sinh(\mu_1)}\right]$$

$$\frac{b_2+b_3}{b_2-b_3}(a_2-a_3) = \frac{\mu_1}{tanh(\mu_1)},$$

and

$$2v_2v_3\left(\frac{a_2-a_3}{b_2-b_3}\right) = \frac{2v_2v_3}{v_3^2}\left(\frac{\mu_1}{b_2/b_3-1}\right) \; ; \quad \left[b_k = v_k^2\right]$$

$$= 2\mu_1\left[\frac{(v_2/v_3)}{(v_2/v_3)^2-1}\right] = 2\mu_1\left(\frac{e^{\mu_1}}{e^{2\mu_1}-1}\right) = \frac{\mu_1}{\frac{1}{2}e^{-\mu_1}\left[e^{2\mu_1}-1\right]} = \frac{\mu_1}{\frac{1}{2}\left[e^{\mu_1}-e^{-\mu_1}\right]}$$

$$2v_2v_3\left(\frac{a_2-a_3}{b_2-b_3}\right) = \frac{\mu_1}{sinh(\mu_1)},$$

it is confirmed that

$$\left.\begin{aligned} \frac{b_i+b_j}{b_i-b_j}(a_i-a_j) &= \frac{\mu_k}{tanh(\mu_k)} \\ 2v_iv_j\left(\frac{a_i-a_j}{b_i-b_j}\right) &= \frac{\mu_k}{sinh(\mu_k)} \end{aligned}\right\} \; ; \quad (i,j,k) \in perm(1,2,3),$$

resulting in the simplified $\mathbf{H}$-tensor representations

$$[\![\underset{\sim}{\mathbf{H}}_1]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[1,1,1,\frac{\mu_1}{tanh(\mu_1)},\frac{\mu_2}{tanh(\mu_2)},\frac{\mu_3}{tanh(\mu_3)}\right],$$

$$[\![\underset{\sim}{\mathbf{H}}_2]\!]_{{}^s\mathbf{A}_\alpha} = diagonal\left[1,1,1,\frac{\mu_1}{sinh(\mu_1)},\frac{\mu_2}{sinh(\mu_2)},\frac{\mu_3}{sinh(\mu_3)}\right].$$

Analysis of the hyperbolic coefficient functions (see Fig. 1) reveals that both are symmetric and strictly positive. This guarantees that the tensors

$$\underset{\sim}{\mathbf{H}}_k : \mathcal{S} \to \mathcal{S} \; ; \quad k = 1,2$$

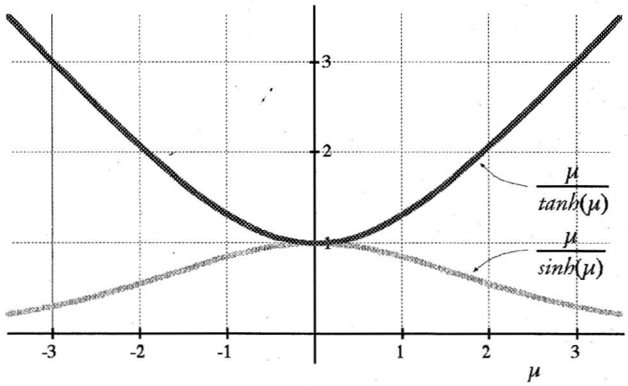

Fig. 1. Plot of hyperbolic coefficient functions

are symmetric, positive definite, bilinear forms on $\mathcal{S} \subset \mathcal{T}$,

$$\left.\begin{array}{l} \mathbf{X}\bullet(\underset{\sim}{\mathbf{H}}_k\mathbf{X}) \left\{ \begin{array}{ll} =0 ; & \mathbf{X}=\mathbf{0} \\ >0 ; & \text{otherwise} \end{array} \right\} ; \quad \mathbf{X}\in\mathcal{S} \\ \underset{\sim}{\mathbf{H}}_k = \underset{\sim}{\mathbf{H}}_k^{\mathrm{T}} \Leftrightarrow \mathbf{X}\bullet(\underset{\sim}{\mathbf{H}}_k\mathbf{Y}) = \mathbf{Y}\bullet(\underset{\sim}{\mathbf{H}}_k\mathbf{X}) ; \quad (\mathbf{X},\mathbf{Y})\subset\mathcal{S} \end{array}\right\} ; \quad k=1,2, \tag{A.51}$$

having the symmetric, positive definite inverse mappings

$$\underset{\sim}{\mathbf{G}}_k \equiv \underset{\sim}{\mathbf{H}}_k^{-1} : \mathcal{S}\rightarrow\mathcal{S} ; \quad k=1,2,$$

corresponding to the eigenbasis representation matrices (A.52)

$$[\![\underset{\sim}{\mathbf{G}}_1]\!]_{{}^{\mathrm{s}}\mathbf{A}_\alpha} = diagonal\left[1,1,1,\frac{tanh(\mu_1)}{\mu_1},\frac{tanh(\mu_2)}{\mu_2},\frac{tanh(\mu_3)}{\mu_3}\right],$$
$$[\![\underset{\sim}{\mathbf{G}}_2]\!]_{{}^{\mathrm{s}}\mathbf{A}_\alpha} = diagonal\left[1,1,1,\frac{sinh(\mu_1)}{\mu_1},\frac{sinh(\mu_2)}{\mu_2},\frac{sinh(\mu_3)}{\mu_3}\right].$$

Making use of the fourth order *identity* tensor

$$\underset{\sim}{\mathbf{I}} : \mathcal{S}\rightarrow\mathcal{S}$$

for which

$$\underset{\sim}{\mathbf{I}}\mathbf{X} = \mathbf{X} \Leftrightarrow [\![\underset{\sim}{\mathbf{I}}]\!]_{{}^{\mathrm{s}}\mathbf{A}_\alpha} = diagonal[1,1,1,1,1,1],$$

it is clear that each of these $\mathbf{H}$ and $\mathbf{G} = \mathbf{H}^{-1}$ tensors can be expressed in the alternative form

$$\left.\begin{array}{l} \underset{\sim}{\mathbf{H}}_k = \underset{\sim}{\mathbf{I}} + \Delta\underset{\sim}{\mathbf{H}}_k \\ \underset{\sim}{\mathbf{G}}_k = \underset{\sim}{\mathbf{I}} + \Delta\underset{\sim}{\mathbf{G}}_k \end{array}\right\} ; \quad k=1,2,$$

in terms of their respective "increment" tensors

$$\left.\begin{array}{l} \Delta\underset{\sim}{\mathbf{H}}_k : \mathcal{S}\rightarrow\mathcal{S} \\ \Delta\underset{\sim}{\mathbf{G}}_k : \mathcal{S}\rightarrow\mathcal{S} \end{array}\right\} ; \quad k=1,2.$$

These *increment* or *delta* tensors are easily confirmed to have the respective eigenbasis representations

$$\left.\begin{array}{l} [\![\Delta\underset{\sim}{\mathbf{H}}_k]\!]_{{}^{\mathrm{s}}\mathbf{A}_\alpha} = diagonal[0,0,0,h_k(\mu_1),h_k(\mu_2),h_k(\mu_3)] \\ [\![\Delta\underset{\sim}{\mathbf{G}}_k]\!]_{{}^{\mathrm{s}}\mathbf{A}_\alpha} = diagonal[0,0,0,g_k(\mu_1),g_k(\mu_2),g_k(\mu_3)] \end{array}\right\} ; \quad k=1,2,$$

expressed in terms of the scalar coefficient functions

$$h_1(\mu) \equiv \frac{\mu}{tanh(\mu)} - 1 = \tfrac{1}{3}\mu^2\left\{1 - \tfrac{1}{15}\mu^2 + \tfrac{2}{315}\mu^4 + \dots\right\} \geq 0,$$
$$h_2(\mu) \equiv \frac{\mu}{sinh(\mu)} - 1 = -\tfrac{1}{6}\mu^2\left\{1 - \tfrac{7}{60}\mu^2 + \tfrac{31}{2520}\mu^4 + \dots\right\} \leq 0,$$
$$g_1(\mu) \equiv \frac{tanh(\mu)}{\mu} - 1 = -\tfrac{1}{3}\mu^2\left\{1 - \tfrac{2}{5}\mu^2 + \tfrac{17}{105}\mu^4 + \dots\right\} \geq 0,$$
$$g_2(\mu) \equiv \frac{sinh(\mu)}{\mu} - 1 = \tfrac{1}{6}\mu^2\left\{1 + \tfrac{1}{20}\mu^2 + \tfrac{1}{840}\mu^4 + \dots\right\} \leq 0, \tag{A.53}$$

which are seen to be of order μ^2 in the neighborhood of their apparent singularity at $\mu=0$. With this, it is a simple matter to verify the eigenbasis expansions

$$\begin{aligned}
\Delta\mathbf{H}_k\mathbf{X} &= (\mathbf{H}_k - \mathbf{I})\mathbf{X} = \mathbf{H}_k\mathbf{X} - \mathbf{X}, \\
\Delta\mathbf{H}_k\mathbf{X} &= h_k(\mu_1)(\hat{a}_2\bullet\mathbf{X}\hat{a}_3)(\hat{a}_2\otimes\hat{a}_3 + \hat{a}_3\otimes\hat{a}_2) \\
&\quad + h_k(\mu_2)(\hat{a}_3\bullet\mathbf{X}\hat{a}_1)(\hat{a}_3\otimes\hat{a}_1 + \hat{a}_1\otimes\hat{a}_3) \\
&\quad + h_k(\mu_3)(\hat{a}_1\bullet\mathbf{X}\hat{a}_2)(\hat{a}_1\otimes\hat{a}_2 + \hat{a}_2\otimes\hat{a}_1) \;;\quad k=1,2, \\
\Delta\mathbf{G}_k\mathbf{X} &= (\mathbf{G}_k - \mathbf{I})\mathbf{X} = \mathbf{G}_k\mathbf{X} - \mathbf{X}, \\
\Delta\mathbf{G}_k\mathbf{X} &= g_k(\mu_1)(\hat{a}_2\bullet\mathbf{X}\hat{a}_3)(\hat{a}_2\otimes\hat{a}_3 + \hat{a}_3\otimes\hat{a}_2) \\
&\quad + g_k(\mu_2)(\hat{a}_3\bullet\mathbf{X}\hat{a}_1)(\hat{a}_3\otimes\hat{a}_1 + \hat{a}_1\otimes\hat{a}_3) \\
&\quad + g_k(\mu_3)(\hat{a}_1\bullet\mathbf{X}\hat{a}_2)(\hat{a}_1\otimes\hat{a}_2 + \hat{a}_2\otimes\hat{a}_1) \;;\quad k=1,2,
\end{aligned} \tag{A.54}$$

for any $\mathbf{X}\in\mathcal{S}$.

With reference to the previous discussion pertaining to the complimentary subspaces $\mathcal{C}(\mathbf{a})$ and $\mathcal{C}^*(\mathbf{a})$ of $\mathbf{a}$, the conditions (A.30), and the coefficient forms (A.53), it is apparent that each of these increment mappings has $[\mathcal{C}(\mathbf{a})\cap\mathcal{S}]$ as its *null* space, and $[\mathcal{C}^*(\mathbf{a})\cap\mathcal{S}]$ as its *range* space. This is formally expressed as

$$\left.\begin{aligned}
&\Delta\mathbf{H}_k : [\mathcal{C}(\mathbf{a})\cap\mathcal{S}] \to \{\mathbf{0}\} \\
&\Delta\mathbf{G}_k : [\mathcal{C}(\mathbf{a})\cap\mathcal{S}] \to \{\mathbf{0}\} \\
&\Delta\mathbf{H}_k : \mathcal{S} \to [\mathcal{C}^*(\mathbf{a})\cap\mathcal{S}] \subset \mathcal{C}^*(\mathbf{a}) \\
&\Delta\mathbf{G}_k : \mathcal{S} \to [\mathcal{C}^*(\mathbf{a})\cap\mathcal{S}] \subset \mathcal{C}^*(\mathbf{a})
\end{aligned}\right\} \;;\quad k=1,2,$$

or alternatively as, (A.55)

$$\left.\begin{aligned}
&\mathbf{X}\in[\mathcal{C}(\mathbf{a})\cap\mathcal{S}] \;\Rightarrow\; \begin{cases} \Delta\mathbf{H}_k\mathbf{X} = \mathbf{0} \;\Leftrightarrow\; \mathbf{H}_k\mathbf{X} = \mathbf{X} \\ \Delta\mathbf{G}_k\mathbf{X} = \mathbf{0} \;\Leftrightarrow\; \mathbf{G}_k\mathbf{X} = \mathbf{X} \end{cases} \\
&\mathbf{X}\in\mathcal{S} \;\Rightarrow\; \begin{cases} [\mathbf{H}_k\mathbf{X} - \mathbf{X}] \in [\mathcal{C}^*(\mathbf{a})\cap\mathcal{S}] \subset \mathcal{C}^*(\mathbf{a}) \\ [\mathbf{H}_k\mathbf{X} - \mathbf{X}] \in [\mathcal{C}^*(\mathbf{a})\cap\mathcal{S}] \subset \mathcal{C}^*(\mathbf{a}) \end{cases}
\end{aligned}\right\} \;;\quad k=1,2.$$

More specifically, after taking note of the identical qualitative behavior of the scalar μ-coefficients in the expansion (A.45) for the composition mapping

$$(2\mathbf{A}_a\circ\mathbf{W}_v)\big|_{\mathcal{S}} : \mathcal{S}\to\mathcal{S},$$

with those in the above expansion (A.54), it is apparent that each of these increment mappings, like $(2\mathbf{A}_a\circ\mathbf{W}_v)$, is one-to-one (and therefore invertible) when restricted to its *range* space $[\mathcal{C}^*(\mathbf{a})\cap\mathcal{S}]$. In addition, these expansions are easily combined to establish the important identity

$$\begin{aligned}
(\mathbf{H}_1 - \mathbf{H}_2)\mathbf{X} &= \Delta\mathbf{H}_1\mathbf{X} - \Delta\mathbf{H}_2\mathbf{X} \;;\quad \mathbf{X}\in\mathcal{S} \\
&= [h_1(\mu_1) - h_2(\mu_1)](\hat{a}_2\bullet\mathbf{X}\hat{a}_3)(\hat{a}_2\otimes\hat{a}_3 + \hat{a}_3\otimes\hat{a}_2) \\
&\quad + [h_1(\mu_2) - h_2(\mu_2)](\hat{a}_3\bullet\mathbf{X}\hat{a}_1)(\hat{a}_3\otimes\hat{a}_1 + \hat{a}_1\otimes\hat{a}_3) \\
&\quad + [h_1(\mu_3) - h_2(\mu_3)](\hat{a}_1\bullet\mathbf{X}\hat{a}_2)(\hat{a}_1\otimes\hat{a}_2 + \hat{a}_2\otimes\hat{a}_1),
\end{aligned}$$

$$\left(\underset{\sim}{\mathbf{H}}_1 - \underset{\sim}{\mathbf{H}}_2\right)\mathbf{X} = \left(2\underset{\sim}{\mathbf{A}}_a \circ \underset{\sim}{\mathbf{W}}_v\right)\mathbf{X} \;\; ; \quad \mathbf{X} \in \mathcal{S}\,, \tag{A.56}$$

which follows as a direct consequence of the relationship

$$k_1(\mu) - k_2(\mu) = \frac{\mu}{tanh(\mu)} - \frac{\mu}{sinh(\mu)} = \mu\, tanh\left(\tfrac{1}{2}\mu\right),$$

derived from $(A.53)_{1,2}$ and (A.46).

References

Casey, J. (1987). (a discussion of) Invariance considerations in large strain elasto-plasticity. *ASME J. Appl. Mech.* **54**, 247-248.

Dashner, P.A. (1979). A finite strain work-hardening theory for rate independent elasto-plasticity. *Int. J. Solids Structures* **15**, 519-528.

Dashner, P.A. (1986). Invariance considerations in large strain elasto-plasticity. *ASME J. Appl. Mech.* **53**, 55-60.

Dashner, P.A. (1986). Large strain inelastic state variable theory. *Int. J. Solids Structures* **22**, 571-592.

Dashner, P.A. (1986). Plastic potential theory in large strain elastoplasticity. *Int. J. Solids Structures* **22**, 593-623.

Dashner, P.A. (1990). Fading memory in an incremental model for elastic fluids. *Int. J. Non-Newtonian Fluid Mechanics* **36**, 305-331.

Drucker, D.C. (1951). A more fundamental approach to plastic stress-strain relations. *Proc. 1st U.S. Natl. Congr. Appl. Mech.*, 487-491. New York: A.S.M.E.

Green, A.E. and Naghdi, P.M. (1965). A general theory of an elastic-plastic continuum. *Arch Rat. Mech. Anal.* **18**, 251-281.

Guo, Z.H., Lehmann, T., Liang, H.Y. and Man, C.S. (1992). Twirl tensors and the tensor equation $\mathbf{AX} - \mathbf{XA} = \mathbf{C}$. *J. Elast.* **27**, 227-245.

Hill, R. (1959). Some basic principles in the mechanics of solids without a natural time. *J. Mech. Phys. Solids* **7**, 209-225.

Hill, R. (1968). On constitutive inequalities for simple materials-II. *J. Mech. Phys. Solids* **16**, 315-322.

Hill, R. and Rice, J.R. (1973). Elastic potentials and the structure of inelastic constitutive laws. *SIAM J. Appl. Math.* **25**, 448-461.

Il'iushin, A.A. (1961). On the postulate of plasticity. *PMM* **25**, 503-507.

Mandel, J. (1971). Plasticité classique et viscoplasticité. *Courses and Lectures, No 97, International Center for Material Sciences*, Udine, Springer, New York.

Scheidler, M. (1994). The tensor equation $\mathbf{AX} + \mathbf{XA} = \Phi(\mathbf{A}, \mathbf{H})$, with applications to kinematics of continua. *J. Elast.*, **36**, 117-153.

Xiao, H., Bruhns, O.T. and Meyers, A. (1997). Logarithmic strain, logarithmic spin and logarithmic rate. *Acta Mech.* **124**, 89-105.